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EXPERIMENTAL HARMONIC MOTION

A MANUAL FOR THE LABORATORY

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PREFACE TO THE SECOND EDITION

THE subject of Harmonic Motion presents difficulties to many students. For some reason they fail to get any real grasp of the principles and in consequence dare not trust themselves to apply them to the simple examples they meet with in practical physics, even in those cases where the mathematical analysis is quite elementary. The present little volume is an attempt to meet the difficulty. The simplest parts of the theory of Harmonic Motion are considered in Chapter I. In Chapter II descriptions are given of a number of experiments which illustrate the principles of the subject. Where necessary, the theory of Chapter I is extended to meet the problem in hand. In each case the method has been found by experience to be such that a serious student can rely upon obtaining a result which he will feel is a satisfactory reward of not more than about two hours' work. In some cases it would be possible to devise arrangements which would secure greater accuracy, but in my class at the Cavendish Laboratory we have to be content with what may be described as Rapid Physics. We teach Mechanics, Heat and Light in the same rooms so that nearly all the apparatus has to be such that it can be readily moved from place to place. I have tried to design the apparatus so that the physical realities may conform as closely as I could make them, under the existing laboratory limitations, to the ideal conditions contemplated in the mathematical theory.

The volume concludes with a few Notes dealing with some points in the mathematical theory of the subject.

This second edition differs from the first only by the addition of EXPERIMENT 15. Here I must acknowledge the help of Dr G. T. Bennett, F.R.S.

In the preface to the Manual on "Experimental Elasticity," published in 1908, I expressed the hope that an "Experimental Optics" would be published in a few months and that, if life and health were given me, this might be followed by some volumes on other parts of physics. But in 1910 I experienced a severe nervous breakdown and I was absent from Cambridge till October 1911.

Since then, except for two years' absence, 1917 to 1919, at the Royal Aircraft Establishment during the war, I have carried on the class, a work making great demands upon a demonstrator, since there have been sometimes over 50 students working in the class at the same time. Those who have "broken down" will perhaps understand that this work of teaching was a sufficient excuse for some delay in the appearance of the Experimental Optics

As the effort required to take up again all the threads of the partially written Experimental Optics and to complete the book would have been considerable, I decided, in making a fresh start in 1914, to take the rather easier course of publishing the work done in my class in Experimental Harmonic Motion. I still hope that the Optics is merely delayed; the manuscript is now nearly completed, and part of the book is in type.

To make the work done in my class available to some extent to other teachers and students, I have in recent years communicated to the *Proceedings of the Cambridge Philosophical Society* accounts of several experiments in Optics and other parts of physics. Some other optical experiments are described in Vol. II. of the *Proceedings of the Optical Convention*, 1912.

I have authorised Messrs W. G. Pye & Co., of Cambridge, to supply apparatus made to my designs.

I have to thank Mr G. Stead, of Clare College, for very efficient help in the preparation of this volume from the manuscripts used in my practical class. Mr J. R. Airey, of St John's College, assisted in 1905 in some of the preliminary work, and my wife has also helped.

I cannot end this preface without expressing my thankfulness for the kindness and consideration of those who have assisted me in the teaching in my class and for the enthusiasm and friendship of the students. Above all, I must give thanks to God for giving me the restoration of health that has enabled me to write this book.

G. F. C. SEARLE.

8 *April* 1922

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CHAPTER I

ELEMENTARY THEORY OF HARMONIC MOTION

1. Introduction. In experimental physics, it very often happens that the system, whose motions are observed, performs oscillations and it is not difficult to give practical reasons why this type of motion occurs so much more frequently than any other type.

To begin with, it is but seldom we can observe a body moving uniformly along a straight line. The slow fall with a limiting velocity of a small sphere through a viscous liquid is one instance, but it would be difficult to name many more.

When a body is falling freely or is under the action of a "diluted" gravitational action, as when a sphere rolls down an inclined plane, or two bodies move in an Atwood's machine, the motion suffers uniform acceleration in a straight line, but it is perhaps only by means of gravity that uniform acceleration in a straight line can be obtained. Uniformly accelerated motions have the practical disadvantage that the interval of time to be measured is short, when the motion is limited to a few metres, as it is in an ordinary room, unless the acceleration is very slow, in which case the effects of disturbing forces may rival the effects we wish to study.

When a body is compelled to turn about a fixed axis, the space required for the movement is comparatively small, so that one of the objections to uniform or uniformly accelerated motion in a straight line does not apply. In the case of uniform angular velocity, a supply of power is required to maintain the motion, as when the disc of a siren is driven by an electromotor. In such

a case we are, however, generally but little concerned with the dynamics of the apparatus, our attention being confined to producing a uniform rotation somehow.

To produce uniform angular acceleration in the absence of friction, a couple of constant magnitude would be necessary, but such a couple it would be very difficult, if not impossible, to produce.

The reader will thus perceive that he is likely to meet with very few instances of uniform or uniformly accelerated motion either along a straight line or round a fixed axis.

When a vibratory motion is substituted for one in which the movement is always in one direction, a great advantage is at once gained. For now, even in the case of rectilinear motion, only a comparatively small space is required; and in both rectilinear and angular motions, although the time of one vibration may be small, it can be found with considerable accuracy by observing the time occupied by a large number of vibrations, as can be done if the vibrations die only slowly away. In most cases, further, the time of vibration is practically independent of the amplitude of vibration, so long as the amplitude is "small."

2. Harmonic motion. On a circle with O (Fig. 1) for its centre, take a point P and draw a perpendicular PN from P upon any diameter AOA'. Then, if P move round the circle with uniform angular velocity, the point N will move backwards and forwards along AOA' in a definite manner, and the motion of N is called harmonic. The length OA is called the amplitude of the oscillation, and the time occupied by N in going from A to A' and back to A is called the time of a complete vibration, or the periodic time.

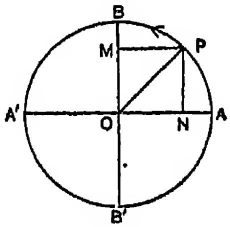

Fig. 1.

Since N is the foot of the perpendicular PN, the velocity and acceleration of N along AOA' are equal to the components, parallel to AOA', of the velocity and acceleration of P.

Let OP revolve in the counter-clockwise direction, let ω radians per second be the angular velocity of OP, let the radius

of the circle be r cm., and let $ON = x$ cm., OA being the positive direction of x. Then, if t seconds be the time since P was last at A, the angle POA is ωt radians. Hence we have

$$x = ON = r \cos \omega t, \quad \ldots\ldots \ldots \quad \ldots \ldots \quad (1)$$

and thus x is proportional to $\cos \omega t$. If PM be drawn perpendicular to OB, where OB is perpendicular to OA, the motion of M will be harmonic also and we shall have

$$y = OM = r \sin \omega t. \ldots\ldots \quad \ldots\ldots\ldots \quad (2)$$

The functions $\cos \omega t$ and $\sin \omega t$ occur in the theory of the vibrations of stretched strings, and it is from the connexion of such strings with the musical scale that the use of the adjective *harmonic* has been extended to the motion of a point whose displacement is proportional to $\cos \omega t$ or $\sin \omega t$.

3 **The velocity.** The length of arc passed over by P in one second is r times the angle turned through by OP in one second, and hence, if v cm sec.$^{-1}$ be the velocity of P along the circumference of the circle,

$$v = r\omega. \ldots \quad \ldots \quad \ldots\ldots\ldots\ldots \quad \ldots(3)$$

Since this velocity is perpendicular to OP, its component parallel to OA is $-r\omega \sin POA$, and thus, if u be the velocity of N along OA,

$$u = -r\omega \sin \omega t = -v \sin \omega t. \quad \ldots\ldots\ldots \ldots(4)$$

4. **The acceleration.** Since u is the rate at which x increases with the time, the rate of increase of $r \cos \omega t$ is $-r\omega \sin \omega t$. Writing $\omega t + \frac{1}{2}\pi$ for ωt in this expression, and multiplying by ω, we see that the rate of increase of $r\omega \cos (\omega t + \frac{1}{2}\pi)$ or of $-r\omega \sin \omega t$ is $-r\omega^2 \sin (\omega t + \frac{1}{2}\pi)$ or $-r\omega^2 \cos \omega t$. But the rate of increase of u is the acceleration of N, and hence, if f cm. sec.$^{-2}$ be the acceleration of N in the direction OA,

$$f = -r\omega^2 \cos \omega t = -\omega^2 x. \ldots \ldots\ldots\ldots\ldots(5)$$

From this equation it will be seen that, when x is positive, f is negative and *vice versa*.

Hence, when a point moves with harmonic motion along a straight line, its acceleration is always directed towards the centre O, and is proportional to its distance from O.

Since, by (3), $\omega = v/r$, we have $r\omega^2 = v^2/r$, and thus

$$f = -\frac{v^2}{r}\cos \omega t. \qquad \text{......(6)}$$

5 Application of the calculus. The velocity and the acceleration of the moving point N (Fig. 1) can be readily found by the use of the calculus.

Since, by (1), $x = r\cos\omega t$,

we have $u = dx/dt = -r\omega \sin\omega t$

and $f = d^2x/dt^2 = du/dt = -r\omega^2 \cos\omega t$,

so that $f = -\omega^2 x$,

as found in § 4.

6. Acceleration of a point in uniform circular motion. The acceleration of N may also be deduced from the acceleration of a point moving uniformly round a circle.

Let P (Fig. 2) be a point moving round a circle of radius r with uniform velocity v, the angular velocity of the radius OP being ω. When the point is at P, it is moving along the tangent PT with the velocity v, and when it is at P', it is moving along the tangent $P'T'$ with the velocity v. If t be the time of describing PP', the angle POP' is equal to ωt. Now the velocity of the moving point at P' can be resolved into $v\sin\omega t$ parallel to PO, and $v\cos\omega t$ parallel to PT. In the time t, the point has gained the velocity $v\sin\omega t$ parallel to PO, and hence, if a_{av} be its average acceleration parallel to PO,

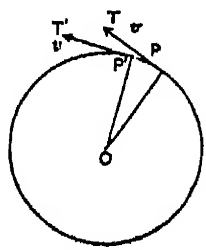

Fig. 2.

$$a_{av} = \frac{v\sin\omega t}{t} = v\omega\frac{\sin\omega t}{\omega t}.$$

As t and ωt approach zero, the average value, during the journey from P to P', of the acceleration of the point parallel to PO, approaches a limiting value, which is its actual acceleration a in the direction from P to O when the point is at P, and thus the acceleration of the point at P is given by the limiting value which a_{av} approaches as t approaches zero. Since $\sin\omega t/\omega t$ approaches

the limiting value unity as ωt approaches zero, the acceleration in the direction PO, of the point when at P, is given by

$$\alpha = v\omega. \qquad \text{.......(7)}$$

We must now see whether the point, as it passes through P, has or has not any acceleration parallel to the tangent PT. If β_{av} be the average acceleration of the point parallel to PT,

$$\beta_{av} = (1/t)(v \cos \omega t - v) = -(2v/t) \sin^2 \tfrac{1}{2}\omega t$$

$$= -v\omega \frac{\sin \frac{1}{2}\omega t}{\frac{1}{2}\omega t} . \sin \tfrac{1}{2}\omega t,$$

and this approaches the limiting value zero when t approaches zero. Thus, when the moving point is at P, it has no acceleration along the tangent at P.

Hence, when a point moves uniformly round a circle, the only acceleration which it has is towards the centre at each instant.

Since $v = r\omega$, the acceleration α can be expressed by any of the three following formulae:—

$$\alpha = v\omega \qquad \text{..(8)}$$

$$= v^2/r \qquad \text{..(9)}$$

$$= \omega^2 r. \qquad \text{.......(10)}$$

We can now deduce the acceleration of N (Fig 1) from that of P. Since the acceleration of P is $\omega^2 r$ or $\omega^2 . PO$ in the direction PO, it follows, from the triangle of accelerations, that the acceleration of P parallel to AO is $\omega^2 . NO$ or $\omega^2 x$ *towards* O. But the acceleration of N is equal to the component of the acceleration of P parallel to AO, and thus f, the acceleration of N in the *positive* direction OA, is given by

$$f = -\omega^2 x. \qquad \text{......(11)}$$

7. The periodic time. As the point P goes round the circle, the point N (Fig. 1) oscillates along AOA' and the times of a complete oscillation of N and of a complete revolution of P are equal. Hence, if the time of a complete oscillation of N, or the periodic time of N, be T seconds, the radius OP describes the angle 2π radians in a time T seconds when moving with the angular velocity ω radians per second.

Hence $$T = \frac{2\pi}{\omega}. \qquad \text{......(12)}$$

Since ω^2 is equal to the acceleration which the point has towards O when x, the displacement, is unity, this result can be written

$$T = \frac{2\pi}{\sqrt{\text{acceleration for unit displacement}}}. \quad \ldots\ldots(13)$$

If a starting point N_0 be chosen on AA' and if at a given instant N is moving through N_0 in a given direction, it is clear that the interval, which elapses before N is again moving through N_0 in the same direction (for the first time), is independent of the position of N_0.

8. Isochronism. The radius of the auxiliary circle does not appear in the formula for the periodic time and hence T is independent of the amplitude. The extent of the oscillation has therefore no influence upon the time of a complete oscillation. In consequence of this property, which is obviously of great importance, the vibrations are called *isochronous*.

9. Summary of results. The results we have obtained may be restated as follows:—If a point N moving along a straight line have an acceleration μx towards a fixed point O on this line, where x is the distance of N from O, the acceleration when there is unit displacement is μ. Hence, by §§ 7, 8, the point performs isochronous vibrations in the time T, where

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{\sqrt{\text{acceleration for unit displacement}}}.$$

If T be observed, the value of μ can be found from the equation

$$\mu = \frac{4\pi^2}{T^2}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(14)$$

10. Differential equation of harmonic motion. The time of oscillation of a point moving along the axis of x and having the acceleration $-\mu x$ can be readily deduced by the calculus. For the equation of motion

$$\frac{d^2x}{dt^2} = -\mu x \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots(15)$$

is satisfied by

$$x = r\cos\mu^{\frac{1}{2}}t + s\sin\mu^{\frac{1}{2}}t, \quad \ldots\ldots\ldots\ldots\ldots(16)$$

where r and s are any constants. If the origin of time be so chosen that, when $\mu^{\frac{1}{2}}t = \frac{1}{2}\pi$, $x = 0$, we have $s = 0$, and thus

$$x = r\cos\mu^{\frac{1}{2}}t, \quad \text{...... ... .(17)}$$

which agrees with the value found in § 2, if $\omega^2 = \mu$. The velocity u is given by

$$u = dx/dt = -r\mu^{\frac{1}{2}}\sin\mu^{\frac{1}{2}}t + s\mu^{\frac{1}{2}}\cos\mu^{\frac{1}{2}}t, \quad \text{........(18)}$$

and, when $s = 0$,

$$u = -r\mu^{\frac{1}{2}}\sin\mu^{\frac{1}{2}}t, \quad \text{. ....(19)}$$

which agrees with the result in § 3.

Whatever the values of r and s, the velocity, dx/dt, and the displacement, x, both go through complete cycles in the time $2\pi/\sqrt{\mu}$, since, when t increases by $2\pi/\sqrt{\mu}$, the quantity $\mu^{\frac{1}{2}}t$ increases by 2π. Hence the periodic time is $2\pi/\sqrt{\mu}$.

It should be noticed that this solution is applicable to any coördinate and is not limited to the case in which a point moves along a straight line. Thus, if ϕ be any coördinate which fixes the position of the body and if there be a restoring acceleration of the corresponding type of the amount $\mu\phi$, the equation of motion will be

$$\frac{d^2\phi}{dt^2} = -\mu\phi, \quad \text{.......................(20)}$$

and the periodic time will be $2\pi/\sqrt{\mu}$ as before.

11. Angular vibrations. In many cases of oscillation, the body, whose motion is under consideration, instead of moving along a straight line, turns about a fixed axis. Here the position of the body is determined by means of a plane containing the axis and fixed in the body, and, as the body vibrates, this plane vibrates through equal angles on either side of a plane containing the axis and fixed in space. If the angular acceleration, i.e. the rate of change of the angular velocity of the moving plane, be proportional to the angle through which it has turned from the reference plane, and if it always tend to bring the moving plane back to the reference plane, then the motion of the body is again called harmonic. We may speak of the acceleration as a restoring acceleration.

If we take a point N moving on a straight line in such a way

that when the moving plane has turned through an angle θ radians from the reference plane in the positive direction, the point N has moved θ cm. in the positive direction from a fixed point O on the line, then the acceleration of N will be numerically equal to the angular acceleration of the moving plane. Hence if the body oscillate under the action of a restoring acceleration $\mu\theta$, the point N will have a restoring acceleration $\mu\theta$ along the straight line. By § 9 the periodic time of N is $2\pi/\sqrt{\mu}$, thus the periodic time of the moving plane is also $2\pi/\sqrt{\mu}$. Since the vibrations of N are isochronous, so are also the vibrations of the body.

Hence, if a body turning about a fixed axis have a restoring acceleration $\mu\theta$, when the body has turned through an angle θ from a zero position, the body will vibrate harmonically and isochronously about that position in the periodic time T, where

$$T = \frac{2\pi}{\sqrt{\mu}} = \frac{2\pi}{(\text{angular acceleration for one radian displacement})^{\frac{1}{2}}}.$$

If T be observed, the value of μ can be found from the equation

$$\mu = \frac{4\pi^2}{T^2}. \qquad \ldots\ldots(21)$$

It will be seen that the above argument applies to any coördinate which fixes the position of any system. If ϕ be such a coördinate and if there be a restoring acceleration corresponding to the coördinate of the amount $\mu\phi$, then the system will have vibrations corresponding to ϕ which are harmonic and isochronous, and have the periodic time $2\pi/\sqrt{\mu}$.

12. Application of dynamics. So far we have been concerned only with kinetics and have merely considered the motion of a system without enquiring *how* that motion has been caused. But, in order to make use of harmonic motion for the determination of some physical quantity Q, we must introduce dynamics and must calculate the acceleration of the system corresponding to some coördinate x in terms of x and Q. If f be the acceleration, we shall find that f is proportional to x (since by supposition the motion is harmonic) and thus we shall be able to write

$$f = -\mu x,$$

where μ is a quantity depending upon Q, but independent of x, at any rate when the amplitude is very small*.

The periodic time T is found by actual observation, and then, by § 9 or § 11, μ can be determined by the equation

$$\mu = \frac{4\pi^2}{T^2},$$

and from this value of μ we can calculate Q.

Instances of this process occur frequently, but the two simple examples given in §§ 13, 14 may aid the reader in applying the process to actual observations.

13. Example (i). A mass M grammes is suspended by a helical spring from a fixed support and the periodic time of the vertical oscillations of M is found to be T seconds. Let us find the restoring force which acts on M, when M is displaced from its equilibrium position through one centimetre, the motion being assumed to be harmonic.

By § 12, the restoring acceleration, when the displacement is x centimetres, is μx cm. sec.$^{-2}$, where

$$\mu = \frac{4\pi^2}{T^2}.$$

Since the mass is M grammes, the restoring force is M times the acceleration and is thus equal to $\mu x M$ dynes. The restoring force is thus proportional to the displacement.

Hence, if F dynes be the restoring force which acts on M when the displacement is one centimetre,

$$F = \mu M = \frac{4\pi^2 M}{T^2} \text{ dynes.}$$

If M be 1066 grammes and T be 1·91 sec., we find that the restoring force for a displacement of one centimetre is

$$F = \frac{4\pi^2 \times 1066}{1{\cdot}91^2} = 11536 \text{ dynes}$$

14. Example (ii). A body suspended by a vertical wire is found to vibrate about the axis of the wire in the periodic time T secs., the moment of inertia of the body about the axis of the

* It can be shown that, except under circumstances which are mathematically conceivable but would not occur in any experiment, if the vibration is isochronous, so that the periodic time is independent of the amplitude, the motion is harmonic.

wire being K gm. cm^2. Let us find the restoring couple which the wire exerts on the body, when the body is displaced from its equilibrium position through one radian.

By § 12, the restoring angular acceleration, when the displacement is θ radians, is $\mu\theta$ radians per second per second, where

$$\mu = \frac{4\pi^2}{T^2}.$$

Since the moment of inertia of the body is K gm. cm.^{2}, the restoring couple is K times the angular acceleration*, and is thus equal to $\mu\theta K$ dyne-cm. The restoring couple is therefore proportional to the displacement.

Hence, if G dyne-cm. be the restoring couple which the wire exerts on the body when the displacement is one radian,

$$G = \mu K = \frac{4\pi^2 K}{T^2} \text{ dyne-cm.}$$

If the body be a disc 20 cm. in diameter, with its axis vertical, having a mass of 275 grammes, K is $\frac{1}{2} \times 275 \times 10^2$, or 13750 gm. cm.^{2}, and thus if T be 1·83 seconds, we find that the restoring couple, when the displacement is one radian, is

$$G = \frac{4\pi^2 \times 13750}{1{\cdot}83^2} = 1{\cdot}6209 \times 10^5 \text{ dyne-cm.}$$

15. Systems with one degree of freedom. In many cases a vibrating system has only one degree of freedom. By this we mean that the configuration of the system is known as soon as a single quantity, which we call a coordinate, is known. As an

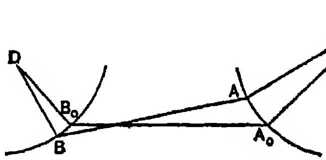

Fig. 3.

example, consider a uniform bar AB (Fig. 3) suspended by two strings from the fixed points C, D. If the system be displaced from its equilibrium position A_0B_0, *the strings remaining in a vertical plane,* the points A, B move on the circles AA_0, BB_0

* *Experimental Elasticity*, Note III.

described about C and D as centres, and the configuration of the system is completely determined when ϕ, the angle ACA_0, is known Since the configuration is known as soon as ϕ is known, ϕ is the coordinate corresponding to the single degree of freedom which the system possesses under the circumstances*.

16. Potential and kinetic energies. If a system having one degree of freedom be displaced from a position of *stable* equilibrium, and be then set free, it will vibrate about that position. Let x be the coordinate corresponding to the single degree of freedom, and let the origin of x be so chosen that x is zero when the body is in the equilibrium position. Let u be the rate at which x increases with the time, thus $u = dx/dt$.

Let the potential energy of the system be taken as zero in the equilibrium position, i e. when x is zero, and let the potential energy when the system is displaced from that position be U. Then U will be a minimum when $x = 0$. Hence, if we construct a curve OP (Fig. 4) showing how U depends on x, the curve must be concave towards the positive direction of U. If we keep within a small enough distance of the origin, the curve will not differ sensibly from a circular arc. But in the case of the circle, PN is proportional to ON^2, when ON is very small, and thus, when x is very small, U will be proportional to x^2. Hence we may write

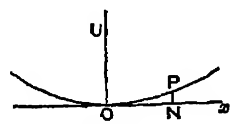

Fig. 4.

$$U = \tfrac{1}{2}\alpha x^2, \qquad \qquad (22)$$

where α may be treated as a constant when x is small enough.

* In reality the rod has four degrees of freedom. The point A moves on the sphere which has its centre at C and to define the position of A requires two coördinates—say, the latitude and the longitude. When A is fixed, the point B can describe a circle upon the sphere which has its centre at D, and one more angular coördinate is required to define the position of B on that circle. These three coördinates completely fix the position of the *straight line* AB. But a *rod* can rotate on its axis and hence, to define its position completely, one more coördinate is required. The total number of coördinates or of degrees of freedom is therefore four There are two constraints, viz. the constancy of the two radii CA, DB, and the number of degrees of freedom added to the number of constraints makes a total of six, the number of degrees of freedom of a rigid body in space. When the points A, B are constrained to move in the vertical plane through CD and the rod is not allowed to rotate on its axis, the degrees of freedom are reduced to one.

Since the velocity of every particle of the system is proportional to u for any configuration, it follows that, if the kinetic energy* of the system be T, we can write

$$T = \tfrac{1}{2}\beta u^2, \qquad \text{(23)}$$

where β is independent of u, though it may depend upon x.

Now, in all ordinary vibrating systems, the velocity of every particle will be *finite* when u is *finite*, and hence, if x change from zero by an infinitesimal amount, the change of configuration of the system will be everywhere infinitesimal.

We may therefore conclude that, when x is restricted to a small enough range, the kinetic energy for a given value of u may be taken as equal to the kinetic energy for the same value of u when x is zero. Hence, when x is small enough, we may treat β as a constant and independent of both x and u.

Thus we see that, in the case of any system which has one degree of freedom and performs small oscillations about a position of stable equilibrium, the potential and kinetic energies can be written

$$U = \tfrac{1}{2}\alpha x^2, \qquad \text{(24)}$$

$$T = \tfrac{1}{2}\beta u^2, \qquad \text{(25)}$$

where α and β are constants.

17 Periodic time. We shall now investigate the motion of a frictionless system for which the potential and kinetic energies are expressed in terms of x and u respectively by equations (24) and (25).

Since the system moves without friction, the law of the Conservation of Energy requires that the sum of the potential and kinetic energies shall remain constant during the motion.

Suppose that, during the motion of the system, x changes to x' and u to u' during the small time τ. Since the total energy $U+T$ remains constant we have

$$\tfrac{1}{2}\alpha x'^2 + \tfrac{1}{2}\beta u'^2 = \tfrac{1}{2}\alpha x^2 + \tfrac{1}{2}\beta u^2,$$

or

$$\tfrac{1}{2}\beta (u'^2 - u^2) = -\tfrac{1}{2}\alpha (x'^2 - x^2),$$

or

$$\tfrac{1}{2}\beta (u' - u)(u' + u) = -\tfrac{1}{2}\alpha (x' - x)(x' + x). \qquad \text{(26)}$$

* The symbol T has been used for kinetic energy in accordance with the usual custom of books on dynamics. It is unfortunate that a quantity which is not a time should have T for its symbol. The only remedy is to use distinguishing marks when there is any risk of confusion.

If f be the rate of increase of u or the acceleration corresponding to the coördinate x, then, when τ is infinitesimally small,

$$\frac{u'-u}{\tau}=f, \quad \frac{x'-x}{\tau}=u,$$

so that

$$(u'-u)\,u=(x'-x)f.$$

Further, when $u'-u$ and $x'-x$ become infinitesimal, we may put u for $\frac{1}{2}(u'+u)$ and x for $\frac{1}{2}(x'+x)$. Hence (26) becomes

$$\beta f=-\alpha x,$$

or

$$f=-\frac{\alpha}{\beta}x. \qquad\qquad (27)$$

Thus the acceleration tends to restore the system to its position of equilibrium, and is proportional to the displacement x. Hence, by § 11, the motion is harmonic, and the periodic time T_0 is given by

$$T_0=2\pi\sqrt{\frac{\beta}{\alpha}}. \qquad\qquad (28)$$

Since the kinetic energy corresponding to unit value of u is $\frac{1}{2}\beta$ and the potential energy corresponding to unit value of x is $\frac{1}{2}\alpha$, we can write (28) in the form

$$T_0=2\pi\sqrt{\frac{\text{kinetic energy for unit velocity}}{\text{potential energy for unit displacement}}}. \qquad (29)$$

The formulae (28) and (29) are of great importance and should be thoroughly understood by the student.

18. Application of the calculus. The matter can be presented in a briefer form by means of the calculus. Thus, since $T+U$ is constant during the vibration,

$$\frac{dT}{dt}+\frac{dU}{dt}=0,$$

or

$$\beta u\frac{du}{dt}+\alpha x\frac{dx}{dt}=0.$$

But

$$\frac{du}{dt}=\frac{d^2x}{dt^2}, \quad \text{and} \quad \frac{dx}{dt}=u,$$

and hence

$$\frac{d^2x}{dt^2}=-\frac{\alpha}{\beta}x.$$

Thus, by § 11, the periodic time T_0 is given by

$$T_0=2\pi\sqrt{\frac{\beta}{\alpha}}.$$

19. Energy formulae. The discussion in § 16 can hardly be considered as furnishing a *proof* of the formulae (24) and (25) for U and T, but merely as giving reasons for expecting that U and T will have those forms if x lies within a very small range of values In any given case, we *must* calculate the actual values of U and T and then we shall be able to see how closely the expressions for U and T approximate to the forms suggested by the discussion in § 16.

As an example, let us consider the rolling bar of EXPERIMENT 13. Here the potential energy U is given in terms of the angular coordinate θ by

$$\begin{aligned} U &= mg\{(r+h)(\cos\theta - 1) + r\theta\sin\theta\} \\ &= mg\{\tfrac{1}{2}(r-h)\theta^2 - (\tfrac{1}{8}r - \tfrac{1}{24}h)\theta^4 + \ldots\}. \end{aligned}$$

When θ is infinitesimal, we have

$$U = \tfrac{1}{2}mg(r-h)\theta^2$$

which is of the form suggested in § 16

The kinetic energy T is given in terms of the angular velocity ω, where $\omega = d\theta/dt$, by

$$T = \tfrac{1}{2}m(k^2 + h^2 + r^2\theta^2)\omega^2,$$

and thus T depends not only upon ω but also upon θ. But, when θ is infinitesimal, $r^2\theta^2$ is to be neglected in comparison with $k^2 + h^2$, and then the kinetic energy becomes

$$T = \tfrac{1}{2}m(k^2 + h^2)\omega^2,$$

which, also, is of the form suggested in § 16.

CHAPTER II

EXPERIMENTAL WORK IN HARMONIC MOTION

20. **Introduction.** In this chapter descriptions are given of a number of experiments in which the laws of harmonic motion are put to the test. To obtain the close agreement between theory and experiment which is possible in spite of the simple character of the apparatus employed, some skill and much care are required, and thus the series of experiments provides a training which may be of much value to the student. One of the greatest obstacles to the satisfactory progress of students is the psychological difficulty of keeping the attention fixed on the work in hand, and, until this difficulty has been overcome, their work will always be liable to error, whatever branch of physics they study The following experiments depend for their success upon accurate observations of periodic times and thus provide a series of exercises which should be worked through very conscientiously by any student who finds difficulty in such observations. A large number of students suffer from this disability, some of them being persons of mathematical ability and of good general intelligence. No student need be ashamed if he finds himself in this class. But he should make every effort to acquire the art of correct timing, since its acquisition will help to give him a feeling of confidence in his own work. Till that feeling is attained, he will always work at a great disadvantage.

21. **Time-pieces.** The object of any timing operation is to compare an interval of time with the period of revolution of the earth on its axis. Since a direct comparison would be inconvenient, a time-piece is used as an intermediary and the going of this time-piece is tested either directly by astronomical observations, as in

examination by aid of a lens of the motion of the seconds hand of the stop-watch during the first second after the button is pressed will leave little doubt that the uncertainty exists.

If a time-piece is used which cannot be started or stopped, the time must be read off from the hand while it is in motion. The error involved in each reading is probably greater than when a stop-watch is used, but, if the method described in § 26 and illustrated in § 27 is carried out carefully over a sufficient interval, very good results may be obtained. It will be very good practice for the student to accustom himself to use an ordinary watch fitted with a seconds hand for this purpose.

22. Defective centering. Some clocks and watches have a very annoying defect. If the minute hand is made to agree with the seconds hand when the former is at some particular minute mark, the agreement does not persist as the watch continues to run. The discrepancy will vanish after an hour and again after two, three...hours. In some cases the discrepancy may amount to as much as half a minute, and then the clock is very unsatisfactory, since there is sometimes a doubt as to which of two minutes is to be taken as the correct reading. The error could arise from a badly divided dial, but in most cases it is probably due to a want of coincidence between the axis of the minute hand and the centre of the dial.

In Fig. 5, let O be the centre of the dial, C the centre about which the minute hand CP turns. Let OCA, the radius of the dial, be a and let $OC = h$. Then, if $ACP = \theta$ and $AOP = \phi$, we have $CPO = \theta - \phi$. Now,

$$\sin(\theta - \phi) = (h/a)\sin\theta,$$

or, when $\theta - \phi$ is small, as it is when h/a is small,

$$\theta - \phi = (h/a)\sin\theta.$$

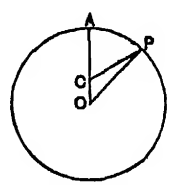

Fig. 5.

Hence the greatest numerical value of $\theta - \phi$ is h/a radians, and this occurs when OP is at right angles to OA. A one minute space subtends an angle of 6° at the centre of the dial, and hence, if the greatest discrepancy between the minute and seconds hands is half a minute, the greatest value of $\theta - \phi$ is 3° or $3/57{\cdot}3$ radians. Then

$$h/a = 3/57{\cdot}3 = 1/19{\cdot}1.$$

In the author's watch the greatest discrepancy is $\frac{1}{8}$ minute of time or $1/57 \cdot 3$ radians and thus $h/a = 1/57 \cdot 3$ Since the radius, a, of the dial is $0 \cdot 8$ inch, we have $h = 0 \cdot 8/57 \cdot 3 = 0 \cdot 014$ inch.

We have taken the most favourable case in which the hands agree when the minute hand points along OA. In the most unfavourable case the hands agree when the minute hand is at right angles to OA, half an hour later the discrepancy will be $2h/a$ radians.

23 **Determination of periodic times.** Since it is essential for accuracy that the vibrating system should continue in motion for several minutes, the system should be firmly supported, so that any dissipation of energy due to the yielding of the support may be as small as possible. A fixed pointer or other indicator is set up so that some mark on the vibrating system is close to the pointer when the system is in its position of equilibrium, and the transit of this mark from *left to right* past the pointer is the signal by which the timing is done. The interval between two successive transits from *left to right* is the periodic time. To avoid confusion, a piece of paper with an arrow drawn on it to indicate the *direction* of the transit may be placed near the pointer.

Some students have the habit of observing the times at which the system reaches the *extremity* of its vibration, but this is an inferior practice, for it is much more difficult to tell when this event occurs than it is to tell when the system passes through its equilibrium position.

When a stop-watch is used, the button is pressed as the mark passes the pointer and "nought" is counted. At the successive transits "one," "two"... are counted. The watch is stopped at some definite transit.

A student sometimes calls "one" when he starts the watch; he stops the watch as he calls "fifty" and though he imagines he has found the time of 50 vibrations he has really found the time of only 49.

When a stop-watch is not available, very good results can be obtained by the method described in § 26 and illustrated in § 27. Another example is given in *Experimental Elasticity*, § 63.

When the periodic time exceeds about two seconds, the mind has time to ramble off to other interests between one count and

the next, and therefore a special effort must be made to keep a watch over the thoughts. It is of assistance to count *out loud.* On account of the difficulty of counting correctly, at least *two* independent observations of any periodic time should be made when a stop-watch is used. If two observations of—say—50 vibrations differ approximately by the periodic time, it is nearly certain that an error of counting has occurred.

EXPERIMENT 1. **Determination of "g" by a simple pendulum.**

24. Theory of simple pendulum. Let O (Fig. 6) be the point of suspension, OA the undisturbed position of the pendulum and OB the position of the pendulum when deflected through an angle θ radians from the vertical. Further, let the mass of the bob be m gm. and the length of the pendulum l cm.

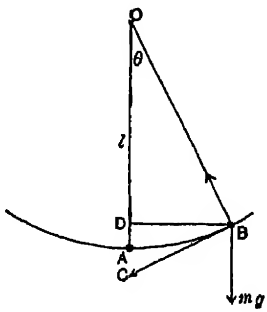

Fig 6.

Then the forces acting on the bob at B are (1) its weight mg dynes acting vertically downwards, (2) the tension of the string. The weight can be resolved into the components $mg \cos\theta$ and $mg \sin\theta$ respectively parallel and perpendicular to OB. The resultant of the former and the tension of the string give the bob the acceleration v^2/l cm. sec $^{-2}$ towards O, where v is the velocity of the bob, and the latter gives the bob an acceleration $g \sin\theta$ cm sec.$^{-2}$ along BC the tangent to the circular path at B. Now the displacement of B, measured along the arc, is $l\theta$ cm. Hence the ratio of the acceleration to the displacement is $g \sin\theta/l\theta$. If the amplitude is very small*, we may put $\sin\theta/\theta = 1$, and then the ratio has the constant value g/l. Hence, by § 11, we have for the time of oscillation

$$T = \frac{2\pi}{\sqrt{\text{acceleration for unit displacement}}} = 2\pi\sqrt{\frac{l}{g}} \text{ secs}, \quad \ldots(1)$$

so that

$$g = \frac{4\pi^2 l}{T^2}. \quad \ldots\ldots\ldots \ldots\ldots\ldots\ldots\ldots(2)$$

* The time of oscillation when the amplitude is finite is investigated in Note II.

We may prove this formula in another way by making use of the result obtained in § 17,

$$T = 2\pi\sqrt{\frac{\text{kinetic energy for unit velocity}}{\text{potential energy for unit displacement}}}.$$

Now the kinetic energy at A is $\frac{1}{2}mv^2$ ergs, where v cm. per sec. is the velocity. Hence the kinetic energy for unit velocity is $\frac{1}{2}m$ ergs. The potential energy of the bob when at B is

$$mg\,.\,AD = mgl\,(1 - \cos\theta) = 2mgl\sin^2\tfrac{1}{2}\theta \text{ ergs.}$$

When θ is small, the potential energy at B may be written $\frac{1}{2}m\,(g/l)\,(l\theta)^2$. Hence, since $l\theta$ cm. is the displacement, the potential energy for unit displacement is $\frac{1}{2}mg/l$ ergs, and thus the periodic time is given by

$$T = 2\pi\sqrt{\frac{\frac{1}{2}m}{\frac{1}{2}mg/l}} = 2\pi\sqrt{\frac{l}{g}}.$$

25. Pendulum with bob of finite size. It must be remembered, of course, that this result applies strictly only to the ideal pendulum considered, and that, as any actual pendulum must have a bob of finite size and a string of finite mass, it is impossible to realize the assumed conditions experimentally. If we treat the whole pendulum as a *rigid* body we find, as in § 49, that, for small oscillations, the periodic time is given by

$$T = 2\pi\sqrt{\frac{K}{mgh}},$$

where K gm. cm.^{2} is the moment of inertia of the pendulum about its axis of suspension, m gm. is the mass of the whole pendulum and h cm. is the distance between the axis of suspension and the centre of gravity of the pendulum.

It may be of interest to find the correction for an actual case. If the mass of the suspending wire be neglected, the centre of gravity of the pendulum is at the centre of the bob, and if the distance of the centre of the bob from the axis of suspension be

l cm., then $h=l$ and $K=K_0+ml^2$, where, for a spherical bob of radius a cm., $K_0=\frac{2}{5}ma^2$ gm cm.²* Then

$$T=2\pi\sqrt{\frac{ml^2+\frac{2}{5}ma^2}{mgl}}=2\pi\sqrt{\frac{l+2a^2/5l}{g}}.$$

Now, if $a=5$ cm. and $l=400$ cm., then

$$\frac{2}{5}\cdot\frac{a^2}{l}=\frac{2}{5}\cdot\frac{25}{400}=0\,025\text{ cm.}$$

That is to say, the effective length of the pendulum is $(400+\cdot025)$ cm. and the error in taking it as 400 cm. is only 1 part in 16,000. (See Note IV.)

26. Experimental details. A convenient form of pendulum consists of an iron sphere, about 10 cm. in diameter, suspended by a steel wire about 400 cm. long and 1 mm. in diameter. The support should be as rigid as possible, for any lateral motion given to it by the vibrations of the pendulum will alter the time of oscillation. It is shown in § 60 that a yielding of the support results in an increase in the effective length of the pendulum by an amount which is equal to the distance through which the weight of the bob, if applied horizontally, would deflect the support

It is often convenient to make the operation of timing in two steps. The periodic time is determined in terms of the indications of a time-piece, and the time-piece itself is rated by the aid of a standard clock. In order to obtain as good a rating as possible, the length of the time during which the time-piece is under observation should be as great as possible. The first comparison of the time-piece with the clock should therefore be made *before* any observations are made on the pendulum. The last comparison should be made as late as the time at the student's disposal permits.

To make a comparison of the time-piece with the standard clock, the time-piece is held near the clock face and the reading of the time-piece is taken when the seconds hand of the clock reaches 60 secs. In order to obtain a more reliable result, the time-piece is read again when the clock hand reaches 15 secs. and again at 30 secs. The mean difference between the readings of the time-piece and the clock for these three observations is then

* *Experimental Elasticity*, Note IV.

found. This process is repeated after the pendulum observations have been made If, however, it is possible to make a series of comparisons of time-piece and clock, the regularity of going of the time-piece can be tested.

The results of each comparison may be entered as follows:

Clock			Time-piece		Clock – Time-piece		
h.	m.	s.	m.	s	h.	m	s.
10	26	0	2	55 6	10	23	4 4
	26	15	3	10 8			4 2
	26	30	3	25·6			4·4
Mean difference = 10 h 23 m 4 3 s.							
11	47	0	83	44 0	10	23	16 0
	47	15	83	59 2			15 8
	47	30	84	14 3			15 7
Mean difference = 10 h. 23 m 15·8 s							

Thus in 1 h. 21 m. or in 81 m the time-piece has lost 15·8 – 4 3 or 11 5 secs. Hence 81 × 60 – 11·5 secs. on the time-piece are equal to 81 × 60 true secs. Thus

$$1 \text{ sec of time-piece} = \frac{81 \times 60}{81 \times 60 - 11{\cdot}5} = \frac{4860}{4848\ 5} \text{ true secs.}$$

The comparison of the time-piece with the pendulum is carried out as follows· The time-piece is placed in a convenient position near the pendulum and, at every fifth transit from left to right of the pendulum past a fixed mark, the time indicated by the time-piece is observed and is recorded upon a sheet of paper prepared for the purpose. After a sufficient number of readings have been taken, the time of the 0th transit is subtracted from that of the 100th, the time of the 5th transit is subtracted from that of the 105th, and so on. In this way we obtain a number of intervals, each corresponding to 100 complete vibrations. With careful work these intervals will agree closely and their mean will furnish a reliable value of the periodic time in terms of one second of the time-piece. The true value of the periodic time is found by multiplying the number of time-piece seconds by the true value of a time-piece second.

The length of the pendulum is then obtained and the value of g is deduced by the aid of the formula (2)

$$g=\frac{4\pi^2 l}{T^2}.$$

The formula should be worked out by *seven* figure logarithms. The value of π^2 is 9·869604, and $\log \pi^2 = 0·9942997$.

As the T in this formula is really the limiting value of the periodic time when the amplitude approaches zero, it is important that in the actual experiment only *small* amplitudes should be used. The effect of finite amplitude is investigated in Note II, and, from the table given there, it will be seen that, when the pendulum swings through 5° on either side of its mean position, the periodic time is greater than that for an infinitely small arc by about 1 part in 2000. As it is the *square* of the time which is involved in the formula for g, this will make a difference of 1 part in 1000 in g itself. Hence the pendulum should not be swinging more than two or three degrees on either side of the mean position when the timing operations are being carried out.

27. Practical example. The observations may be entered as in the following record of an experiment by G. F. C. Searle. Time-piece A was used.

Comparison of time-piece with clock.

Clock			Time-piece			Clock – Time-piece			Mean difference		
h.	m.	s.	h.	m.	s.	h.	m.	s.	h.	m.	s.
10	26	0	0	2	55·6	10	23	4·4	10	23	4·3
		15		3	10·8			4·2			
		30			25·6			4·4			
10	40	0	0	16	53·3	10	23	6·7	10	23	6·7
		15		17	8·3			6·7			
		30			23·3			6·7			
10	51	0	0	27	52·5	10	23	7·5	10	23	7·5
		15		28	7·5			7·5			
		30			22·5			7·5			
11	11	0	0	47	50·2	10	23	9·8	10	23	9·7
		15		48	5·2			9·8			
		30			20·5			9·5			
11	36	0	1	12	45·5	10	23	14·5	10	23	14·4
		15		13	0·7			14·3			
		30			15·7			14·3			
11	47	0	1	23	44·0	10	23	16·0	10	23	15·8
		15			59·2			15·8			
		30		24	14·3			15·7			

From these results we obtain the following table:

Interval from start in minutes of clock	Seconds lost by time-piece
0	0
14	2·4
25	3 2
45	5 4
70	10 1
81	11 5

If these numbers are plotted on a diagram, it will be seen that the rate of the time-piece is irregular. If we take the whole of the 81 minutes, the average rate of loss of the time-piece is 11 5/81 or 0·142 second per minute The comparison with the pendulum occurred in the interval between 10 h. 51 m of the clock and 11 h. 11 m Over this range the time-piece loses 2 2 of its own seconds in 20 minutes of the clock or 0·11 second per minute.

Comparison of pendulum and time-piece.

Transit	Time m	Time s	Transit	Time m	Time s.	Time for 100 vibrations m.	Time for 100 vibrations s.
0	32	0 0	100	38	34·2	6	34·2
5		19 8	105		54 0		34 2
10		39 5	110	39	13 5		34·0
15		59 3	115		33 0		33 7
20	33	19·0	120		53·0		34·0
25		38 5	125	40	12 5		34 0
30		58 4	130		32 3		33 9
35	34	18 2	135		52 2		34 0
40		37 5	140	41	11·8		34·3
45		57 5	145		31·3		33 8
50	35	17 2	150		51·2		34·0
55		36·5	155	42	11·0		34 5
60		57 0	160		30·6		33·6
65	36	16·4	165		50·0		33 6
70		36 0	170	43	10·3		34·3
75		56 0	175		29·7		33 7
80	37	15 5	180		49 4		33 9
85		35·0	185	44	9·2		34 2
90		55 0	190		28 8		33·8
95	38	14 5	195		48·5		34 0
100		34 2	200	45	8·2		34 0

Hence mean time for 100 vibrations = 6 m. 33·99 s = 393·99 seconds of the time-piece.

The time-piece loses 0 11 of its own seconds in one minute of the clock. Thus:

True value of one second of time-piece $= \frac{60}{59\ 89}$ true seconds.

Hence, if T is the true time of one complete vibration,

$$T = 3{\cdot}9399 \times \frac{60}{59{\cdot}89} = 3{\cdot}94714 \text{ seconds.}$$

The length of the pendulum from the point of suspension to the centre of the spherical bob was 388·01 cm. Hence, by (2),

$$g = \frac{4\pi^2 l}{T^2} = \frac{4\pi^2 \times 388{\cdot}01}{3{\cdot}94714^2} = 983{\cdot}19 \text{ cm. sec.}^{-2}.$$

If we had used the rate of loss obtained from the whole of the 81 minutes, viz. 0·142 second of time-piece per minute of clock, we should have had

$$T = 3{\cdot}9399 \times \frac{60}{59{\cdot}858} = 3{\cdot}94925 \text{ seconds,}$$

and then $$g = 982{\cdot}14 \text{ cm. sec.}^{-2}.$$

EXPERIMENT 2. **Harmonic motion of a body suspended by a spring.**

28. Approximate theory. The principles of harmonic motion may be illustrated by an experiment in which a heavy body, suspended from a fixed support by a light helical spring, performs vertical oscillations. We shall first give an account of the approximate theory of the experiment in order that the student may understand what measurements it is necessary to make

Let the total mass suspended from the spring be M gm., and the mass of the spring itself S gm. At present we suppose that S is negligible in comparison with M. When the spring is at rest under the action of the load M, its lower end will be at a definite position which we shall call the standard position. If a small additional mass m gm. be added, the lower end of the spring will be h cm. below the standard position, and, with a suitable spring and a suitable value for M, it will be found by experiment that if, instead of increasing the load from M to $M+m$ we diminish it from M to $M-m$, the lower end of the spring rises h cm. above the standard position. It will also be found that h is proportional to m.

Hence, if the load must be increased from M to M' to stretch the spring by z cm. from the standard position, we have

$$M' = M + \frac{m}{h}z. \quad \text{.........................(1)}$$

Thus, when the displacement of the lower end of the spring is z cm, the upward pull of the spring upon the load is $M'g$ dynes.

Keeping the load at the standard value M, let us now consider the two forces which act upon M when it is performing vertical oscillations and is z cm below its standard position. One force is the weight Mg dynes acting vertically downwards, while the other is the upward pull of the spring. As we are supposing the spring to be without mass, no forces are required to give its various parts the accelerations which they actually have, and hence the upward pull of the spring upon M when it is *in motion*, is equal to the upward pull of the spring when it is *at rest*, the end of the spring being z cm. below the standard position in each case. The upward pull in the latter case has already been found to be $M'g$ dynes, where M' is given by (1).

The acceleration of M can now be calculated. The resultant force is $(M'g - Mg)$ dynes or mgz/h dynes acting upwards, and hence, if the acceleration of M towards its equilibrium position be f cm. sec.$^{-2}$, we have

$$f = \frac{mg}{Mh} z \text{ cm. sec}^{-2}. \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (2)$$

The mass M has thus a restoring acceleration proportional to its displacement, and hence, by §§ 4, 9, its motion is harmonic, and T, the periodic time, is given by

$$T = \frac{2\pi}{(\text{acceleration for unit displacement})^{\frac{1}{2}}}$$

$$= 2\pi \sqrt{\frac{Mh}{mg}} \text{ secs.} \qquad \ldots\ldots\ldots\ldots\ldots\ldots\ldots (3)$$

A more complete theory (§ 30) shows that, when S, the mass of the spring, is small, but not negligible compared with M, the periodic time is no longer given by (3) but approximately by

$$T = 2\pi \sqrt{\frac{(M + \frac{1}{3}S)\, h}{mg}} \text{ secs.} \qquad \ldots\ldots\ldots\ldots (4)$$

The formula (4) indicates that, if we are to compare theory with experiment, we must know M, S, and g and must find the value of h corresponding to any given small mass m. The value of T can then be calculated by (4) and the calculated value can be compared with that found by the use of a watch.

29. Experimental details. A helical spring suitable for this experiment may be made by winding a steel wire, about 8 mm. in diameter, on a cylinder, about 1·5 cm. in diameter, and the unstretched length of the spring may be conveniently about 20 cm. The winding is done in a lathe. An indicator is attached to the lower end of the spring, and below this hangs a pan. The indicator may consist of a plate of brass, about 10 cm. long, 1·5 cm wide, and 3 mm. in thickness, provided with small holes at each end for the attachment of the spring and pan, and with a bolt and nut in the middle for clamping a needle to serve as a pointer. A vertical scale, divided in millimetres, is arranged so that the needle moves along it, and the needle should have a sufficiently fine point to enable readings to be taken to a *tenth* of a millimetre. The general arrangement of the apparatus is shown in Fig 7. The hook or ring by which the spring is suspended should be capable of rotation about a vertical axis, so that the needle may be made to *touch* the scale when the latter is in a convenient position. Care should be taken to see that the scale is vertical, and both spring and scale should be supported as rigidly as possible.

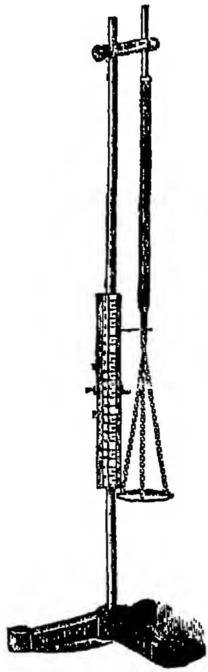

Fig 7.

In performing the experiment, the pan and indicator are detached from the spring and weighed, their mass being P gm. The spring itself must be weighed separately, and its mass S gm. determined. If the load in the pan during the vibration is Q gm., then $P+Q=M$. The mass in the pan is now varied from $(Q-50)$ to $(Q+50)$ gm. by steps of 10 gm., so that m varies from -50 to $+50$ gm. and the reading of the position of rest of the indicator is taken for each load. From these readings the depression of the pointer below its standard position is found for each load. The values of m and h found in this way are plotted on squared paper, and, if the readings have been taken carefully, it will be found that the points obtained lie very approximately on a straight line

passing through the point $m = 0, h = 0$, and thus m/h may be taken as a constant quantity. To determine its value as accurately as possible, the readings of the pointer for $m = -50$ and $m = +50$ gm. are taken several times, the scale being moved after each pair of readings, in order that each pair of readings may be quite independent of the others.

The periodic time can now be calculated from the equation (4)

$$T = 2\pi\sqrt{\frac{(M + \frac{1}{3}S)h}{mg}} \text{ secs.}$$

The value of g is known to within 1 part in 1000, while $(M + \frac{1}{3}S)$ is easily found to the same accuracy, and thus the accuracy of T depends mainly upon that of m/h. If m/h can be found to 1 in 500, the value of T will not differ from the true value by more than 1 in 1000.

The periodic time must next be found by means of a watch. Care must be taken that the value which M has during the determination of T is the same as that used in the formula. Three determinations of T should be made, and in each case at least 100 vibrations should be observed, with a view to finding T to within 1 in 1000. The watch should then be compared with a reliable clock and the value of T should be corrected for any error in the rate of the watch.

30. Approximate correction for the mass of the spring*. When the mass M performs vertical oscillations, the particles of the spring oscillate also, but the *exact* relation between the motion of any part of the spring and the motion of M can only be found by differential equations when the mass of the spring is finite. If, however, S be small compared with M, a method due to Lord Rayleigh enables the periodic time to be written down easily, when, as is here supposed, the spring is uniform when it is unstretched.

When the spring is unstretched, let its length be l cm., so that the mass per unit length is S/l. When the spring carries the standard load M and is at rest, it will be extended, and the upper parts of the spring will be more extended than the lower parts on account of the weight of the spring itself. The position of any

* Students who find § 30 difficult to follow should pass on to § 31.

particle of the spring in this case may be called its standard position. If the spring be still further extended, the tension at every point of the spring will be increased by the *same* amount, and since the ratio of the *increase* of length to the *unstretched* length of every element depends only upon the *increase* of the tension, and not upon the tension before the increase, we see that the displacement of any particle P below its standard position is proportional to x, the distance of P from the fixed end of the spring when the latter is *unstretched*. Hence, if M be displaced through a distance z, the particle P is displaced through the distance zx/l.

When M oscillates, the inertia of the spring comes into play, and the displacement of P from its standard position is no longer zx/l but a more complicated function of x, which the exact theory shows to be $z \sin qx/\sin ql$, where q is a constant*. Thus, if the velocity of M be u, the velocity of P is *not* ux/l but $u \sin qx/\sin ql$, and also the potential energy of the system for a given value of z is not the same as when M was at rest with the same displacement. But Lord Rayleigh's theorem† shows that we shall obtain a close approximation to the periodic time if we calculate the potential and kinetic energies on the assumption that the displacement of P is zx/l and that the velocity of P is ux/l.

The mass of a length dx of unstretched spring is $S\,dx/l$, and hence the kinetic energy of this portion is, on the above assumption,

$$\tfrac{1}{2} \cdot \frac{S\,dx}{l} \cdot \frac{u^2x^2}{l^2} = \tfrac{1}{2} \cdot \frac{Su^2}{l^3} \cdot x^2\,dx.$$

If T, the kinetic energy of the whole system, be $\tfrac{1}{2}au^2$, we have

$$T = \tfrac{1}{2}Mu^2 + \tfrac{1}{2} \cdot \frac{Su^2}{l^3}\int_0^l x^2\,dx.$$

Since $$\int_0^l x^2\,dx = \tfrac{1}{3}l^3,$$

$$T = \tfrac{1}{2}au^2 = \tfrac{1}{2}(M + \tfrac{1}{3}S)u^2.$$

The potential energy, V, is, on the above assumption, equal to the work which is done in moving M infinitely slowly through

* See Note I.

† Lord Rayleigh, *Theory of Sound*, vol. I. § 88, Second Edition.

a distance z from its standard position. When M has been displaced through z cm. from its standard position and is at rest, the upward pull of the spring against it is, by (1), $(M+mz/h)g$ dynes, while the downward pull upon M due to gravity is Mg dynes. Hence the force which must be applied to M to hold it in this position is mzg/h dynes, and the work done by the applied force while the displacement increases from zero to z is z times the average value of the force. Thus, if $V=\frac{1}{2}\beta z^2$,

$$V=\tfrac{1}{2}\beta z^2 = z \cdot \tfrac{1}{2} \cdot \frac{mzg}{h} = \tfrac{1}{2} \cdot \frac{mg}{h} \cdot z^2.$$

We have thus found that the kinetic energy can be expressed in the form $\frac{1}{2}au^2$, and the potential energy in the form $\frac{1}{2}\beta z^2$. Hence, by § 17, the motion is harmonic, and its periodic time, T, is given by

$$T = 2\pi\sqrt{\frac{\text{kinetic energy for unit velocity}}{\text{potential energy for unit displacement}}}$$

$$= 2\pi\sqrt{\frac{\frac{1}{2}(M+\frac{1}{3}S)}{\frac{1}{2}mg/h}} = 2\pi\sqrt{\frac{(M+\frac{1}{3}S)h}{mg}} \text{ secs.,}$$

as stated in § 28.

31. Practical example. The results of an actual experiment are given in the following table.

Load in pan	m	Reading of pointer	h
gm	gm.	cm.	cm
1000	−50	18·14	−2·46
1010	−40	17·65	−1·97
1020	−30	17·16	−1·48
1030	−20	16·67	−0·99
1040	−10	16·18	−0·50
1050	0	15·68	0·00
1060	+10	15·19	+0·49
1070	+20	14·70	+0·98
1080	+30	14·20	+1·48
1090	+40	13·71	+1·97
1100	+50	13·21	+2·47

If the values of m and h, found in this way, are plotted on squared paper, the points are found to lie very approximately on a straight line, so that m/h

may be taken as a constant quantity. To determine the value of m/h as accurately as possible, the following readings were taken —

Scale readings		Difference
$m = -50$ gm.	$m = +50$ gm.	
cm	cm	cm.
17·14	12·21	4·93
17 30	12 38	4 92
17 58	12 65	4 93
18 14	13 21	4 93

The mean difference is 4 93 cm. and thus the mean value of h/m is 4·93/100 or 0 0493 cm per gm. The value of M in this experiment was 1093 0 gm and the value of S was 96 0 gm.

Thus, by (4), the periodic time is given by

$$T = 2\pi\sqrt{\frac{(M+\frac{1}{3}S)\,h/m}{g}} = 2\pi\sqrt{\frac{(1093+96/3)\times 0\ 0493}{981}}$$
$$= 1\ 494 \text{ seconds.}$$

The time of 100 vibrations was found by a stop-watch keeping good time The observed times were 149 5, 149 3, 149 5 secs. The mean value of the periodic time is 1 494 secs., agreeing to four significant figures with the value calculated from the formula.

Experiment 3. Harmonic motion of a rigid body suspended by a torsion wire*.

32. Introduction. This experiment is designed to illustrate the principles of harmonic motion in the case of a rigid body vibrating about a vertical axis under the action of a torsion wire.

33. Theory of experiment. Let a rigid body be suspended from a fixed support by a vertical torsion wire and let the moment of inertia of the body about the axis of the wire be K gm. cm.^{2}. Let the couple required to turn the rigid body through one radian against the torsion of the wire be μ dyne-centimetres. Then, when the angle is θ radians, the couple is $\mu\theta$ dyne-cm., and, when the body is free to move, the angular acceleration of the body is $\mu\theta/K$ radians per sec. per sec.† towards the equilibrium position.

* *Proceedings Camb Phil. Soc.*, vol. XVIII p. 81.

† *Experimental Elasticity*, Note III.

The motion is therefore harmonic and, by § 11, the periodic time is given by*

$$T = \frac{2\pi}{(\text{angular acceleration for one radian displacement})^{\frac{1}{2}}}$$

$$= 2\pi\sqrt{\frac{K}{\mu}} \text{ seconds.} \quad \ldots\ldots\ldots\ldots\ldots\ldots \quad (1)$$

In the experiment, K is found by weighing and measuring the rigid body and μ is found from a series of measurements of the angle through which the lower end of the wire is turned by a series of couples applied statically. The periodic time is calculated by (1) and this time is compared with that which is observed when the body is allowed to vibrate. The agreement between the observed and the calculated values of the periodic time forms a test of the accuracy of the dynamical principles employed.

34. **The vibrating system.** It is essential that the torsion wire should be properly secured (1) to the fixed support and (2) to the vibrating body. This result is best obtained by soldering each end of the wire into a hole drilled along the axis of a cylindrical rod a few centimetres in length and about 0·5 cm. in diameter. One of these rods is secured by a set-screw to the fixed support and the other is secured by a set-screw in a hole drilled in any body which is to be suspended by the wire. These rods are so much stiffer than the torsion wire that small variations in the positions of the points at which the set-screws press upon them make little difference in the couple required to turn the suspended body through one radian against the torsion of the wire. Care should, however, be taken that the set-screws, which fix (1) the vibrating system, and (2) the cylinder shown in Fig. 9, make contact with the rod at nearly the same point. The torsion wires used for this experiment in the author's practical class are of steel and are about 32 cm. in length and 0·175 cm. in diameter.

A convenient rigid body is a rectangular bar (Fig. 8) of length $2L$ cm., of width $2A$ cm., and of depth $2B$ cm. A hole, into which the cylindrical rods attached to the torsion wire fit, is drilled through the bar at right angles to the plane of the edges $2L$, $2A$,

* It must be noted that the μ of § 11 corresponds in the present case not to μ but to μ/K.

as in Fig. 8. The effective mass of the bar, M grammes, should be marked upon the bar. This is the mass of the bar *before* the hole was drilled through it or the set-screw was fitted to it*

The moment of inertia of the bar is calculated by the formula†

$$K = \tfrac{1}{3}M(L^2 + A^2) \text{ gm. cm.}^2 \quad \ldots\ldots\ldots\ldots\ldots(2)$$

The mass of the metal taken out of the hole and the mass of the set-screw are appreciable in comparison with the mass of the bar itself, and hence, if the actual mass of the bar after it has been drilled and fitted with a set-screw were used in formula (2), an appreciable error in K would result. But the hole and the set-screw are so close to the axis of vibration that the *moments of*

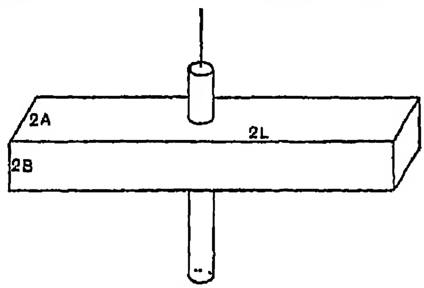

Fig. 8.

inertia of the metal taken from the hole and of the set-screw about the axis are quite inappreciable compared with that of the bar itself and thus the moment of inertia of the bar, as actually used, does not differ appreciably from that given by (2), provided that by M is understood the mass of the bar *before* the hole was drilled through it or the set-screw was fitted to it.

The moment of inertia of the cylindrical rod soldered to the wire is quite negligible in comparison with that of the bar.

One of the heavy compound laboratory stands supplied by W. G. Pye and Co., of Cambridge, forms a convenient support for the upper end of the torsion wire. Whatever support is used should be rigid and free from shake.

The periodic time of the inertia bar should be deduced from two or three observations of the time occupied by at least 100 complete vibrations, and the time-piece should, if necessary, be compared with a reliable clock.

* *Experimental Elasticity*, Note VII. † *Ibid.*, Note IV.

35. Determination of the relation between couple and angle. In the determination of the couple required to twist the lower end of the wire through one radian, the inertia bar is removed from the torsion wire and a cylinder is substituted, as shown in Fig. 9. The couple is applied by means of a thread passing over two ball-bearing pulleys and supporting two small scale-pans, it is convenient to adjust the mass of each pan to be 10 gm. A loop is made in the thread and this loop is passed over the set-screw securing the cylinder to the rod at the end of the torsion wire.

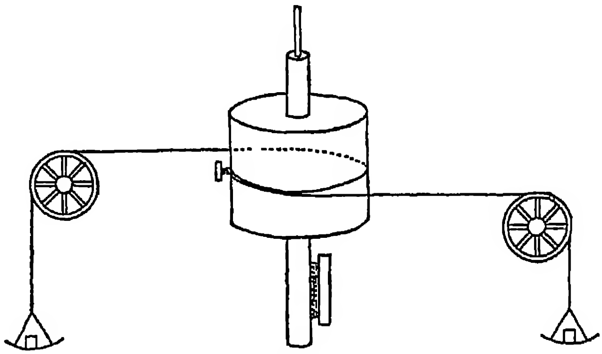

Fig 9.

Care must be taken that the parts of the thread between the cylinder and the pulleys are parallel and horizontal so that, when the two loads are equal, the threads may exert a pure couple on the cylinder Special care must be taken to ensure that the threads are *tangential* to the cylinder.

If the diameter of the cylinder be D cm. and that of the thread be d cm., the distance between the axes of the two threads is $D+d$ cm., and hence, when the load on each thread is m gm., the couple is

$$G = mg(D+d) \text{ dyne-cm.} \quad \ldots\ldots\ldots\ldots\ldots\ldots (3)$$

The angle, θ, through which the couple G causes the cylinder to revolve, is determined by aid of the simple goniometer shown in Fig. 10.

This instrument was designed in conjunction with W. G. Pye and Co. to provide the students at the Cavendish Laboratory with

a means of measuring angles up to about $\frac{1}{4}$ radian (about 14°) with an accuracy of $\frac{1}{1000}$ radian.

The base is formed of a strip of wood furnished at one end with a spherical pivot and at the other with a cross-bar carrying a scale. Angles are measured by means of a moveable arm which turns at one end about the pivot, while the other end moves over the scale on the cross-bar. The optical system consists of a lens fixed to the arm near the pivot and of a fine vertical wire attached to the other end of the arm and adjusted to be in the focal plane of the lens.

The spherical pivot is a phosphor-bronze ball attached to the base by a fitting which allows the distance between the ball and the scale to be adjusted. The ball enters a conical hole turned out of a block of brass attached to the arm. This arrangement destroys three out of the six degrees of freedom of the arm relative to the

Fig. 10.

base. The other end of the arm carries two brass feet which rest upon the cross-bar and thus destroy two degrees of freedom. The remaining degree of freedom allows the arm to turn about an axis through the centre of the ball and perpendicular to the plane of the surface of the cross-bar. This design has the advantage that it is impossible to strain the instrument by lifting it by the moveable arm, for the arm at once comes away from the base.

The scale on the cross-bar is divided into millimetres, and the ball is adjusted so that its centre is 40 cm. from the edge of the scale. The readings are taken by means of a fine wire passing across an opening in the arm and stretched by a spring; the wire is easily replaced if broken. The scale is engine-divided on white metal and is provided with an anti-parallax mirror. For small

angles, one centimetre along the scale corresponds to $\frac{1}{40}$ radian; as the scale can be read to $\frac{1}{100}$ cm., the angle can be read to $\frac{1}{4000}$ radian, or to about $\frac{1}{70}$ degree.

The lens attached to the moveable arm is achromatic and has a focal length of 35 cm. The vertical wire is held in an adjustable frame attached to the arm and is kept tight by a spring, and this frame is adjusted so that the wire is in the focal plane of the lens. The image of a distant point will then fall upon the wire, if the arm be properly directed. If a plane mirror be placed so that the lens lies between it and the wire, the image of the wire formed by two refractions through the lens and one reflection at the mirror may be made to coincide with the wire itself.

When the goniometer is used in mechanical experiments to determine the angle turned through by a body about a vertical axis, a plane mirror is attached to the body and the image of the goniometer wire is made to coincide with the wire itself. If this coincidence is restored after the body has turned, the angle turned through by the body is equal to that turned through by the goniometer arm. To facilitate the adjustment, the short scale which is provided with the instrument is fitted into the frame holding the wire, the divided face of the scale being turned towards the lens. On looking at the wire in the direction of the lens, an inverted image of the scale (formed by the lens and the plane mirror) will be seen crossing the wire. In this use of the instrument, all that is necessary is that the mirror should be nearly vertical and that the rays from the wire, after passing through the lens, should fall upon the mirror. No other adjustments are required, and the centre of the spherical pivot need not lie on the vertical axis about which the body turns.

The goniometer is placed so that its lens is three or four centimetres from the mirror carried by the suspended system.

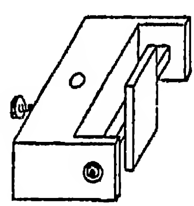

Fig. 11.

In the present experiment, a plane mirror is attached to the suspended system, in the manner shown in Fig. 9, by means of soft wax, or, more conveniently, by means of the simple device shown in Fig. 11, in which the mirror is attached to a horizontal axis and is

thus capable of easy adjustment. The mirror is adjusted so that it is possible to see the inverted image of the small scale attached to the goniometer arm crossing the vertical wire of the goniometer The goniometer is securely fixed so as to be free from shake and from liability to accidental displacement. The *base* is adjusted so that, when the arm is in its central position, the image of the wire coincides as nearly as possible with the wire itself. The *arm* is then adjusted so that the image of the wire coincides exactly with the wire itself, and the reading of the indicating wire on the *edge* of the scale on the cross-bar of the goniometer is taken.

The thread is then attached to the cylinder and is passed over the pulleys, carrying the pans alone; the mass of each pan should be adjusted to be 10 gm. The goniometer arm is then moved until the goniometer wire again coincides with its own image, and the reading on the scale is taken. The load on each end of the thread is then increased by steps of 10 gm. and the observations are repeated for each load. The goniometer arm turns through the same angles as the cylinder, when it is properly adjusted at each stage.

If the reading on the scale of the goniometer for any position of the arm differs from the reading when the arm is in the central position by x cm., and if the distance from the centre of the spherical pivot to the *edge* of the scale be p cm., then the displacement of the arm is θ radians, where $\tan\theta = x/p$. Hence $\theta = \tan^{-1}x/p$. When x/p is small, we have

$$\theta = \tan^{-1}\frac{x}{p} = \frac{x}{p} - \frac{1}{3}\frac{x^3}{p^3} + \frac{1}{5}\frac{x^5}{p^5} - \dots.$$

If x/p is less than 1/20, it will generally be sufficient to put $\theta = x/p$.

When x/p is greater than 1/20 it will be best to calculate the value of $\tan\theta$ and from this to find θ in *degrees*. Bottomley's tables may then be used to find the value of θ in *radians*.

On dividing each value of θ by the corresponding value of m, the result will be very nearly constant, thus showing that the angle is proportional to the couple. The mean value of θ/m is then found and is used in the calculation of μ from the formula

$$\mu = \frac{G}{\theta} = \frac{g\,(D+d)}{\theta/m}.$$

Using this value of μ and the value of K already found from the dimensions of the inertia bar, the time of vibration of the bar is calculated by the formula (1)

$$T = 2\pi\sqrt{\frac{K}{\mu}}.$$

36. Practical example. The observations may be entered as in the following record of an experiment by G. F. C. Searle, Oct. 1906.

Mass of inertia bar $= M = 826$ gm.
Length of inertia bar $= 2L = 37{\cdot}88$ cm.
Width of inertia bar $= 2A = 1{\cdot}60$ cm.
Moment of inertia of bar $= K = \frac{1}{3}M(L^2 + A^2) = 9{\cdot}894 \times 10^4$ gm. cm.^{2}.
Distance from scale to nearer side of pivot $= 39{\cdot}92$ cm.
Diameter of pivot $= 0{\cdot}38$ cm.
Hence $p = 39\,92 + \frac{1}{2} \times 0{\cdot}38 = 40{\cdot}11$ cm.†.
Diameter of cylinder $= D = 3{\cdot}15$ cm.
Diameter of thread $= d = 0{\cdot}03$ cm.

Load gm.	Reading cm.	x cm.	$\tan\theta$	θ degrees	θ radians	$1000\,\theta/m$ radians/gm.
0	15 00	0	0	0	0	
10	14·43	0·57	0·0142	*	0·0142	1·420
20	13·87	1·13	0·0282	*	0·0282	1·410
30	13 29	1 71	0·0426	*	0·0426	1·420
40	12·71	2 29	0·0571	*	0·0571	1·428
50	12·12	2·88	0·0718	4° 6′	0·0716	1·432
60	11·55	3·45	0·0860	4° 55′	0·0858	1·430
70	10 95	4·05	0·1010	5° 46′	0·1007	1·439
80	10·38	4·62	0·1152	6° 34′	0·1146	1·432
90	9·81	5·19	0·1294	7° 22′	0·1286	1·429
100	9·20	5·80	0·1446	8° 14′	0·1437	1·437

* θ put equal to $\tan\theta$ here.

Mean value of $\theta/m = 1{\cdot}428 \times 10^{-3}$ radians per gm.

Hence, $\mu = \dfrac{g(D+d)}{\theta/m} = \dfrac{981 \times 3{\cdot}18}{1{\cdot}428 \times 10^{-3}} = 2{\cdot}185 \times 10^6$ dyne-cm. per radian.

Thus, by (1)

$$T = 2\pi\sqrt{\frac{K}{\mu}} = 2\pi\sqrt{\frac{9{\cdot}494 \times 10^4}{2{\cdot}185 \times 10^6}} = 1{\cdot}338 \text{ secs.}$$

Direct observations with a good stop-watch gave 100 vibrations in 134·2, 133·7, 133·7 seconds. The mean value $T = 1{\cdot}339$ agrees very closely with that deduced from the statical experiments.

† In the goniometer used in this experiment p was not capable of adjustment. In the instrument shown in Fig. 10, p is adjusted to be 40·00 cm.

EXPERIMENT 4. **Study of a system with variable moment of inertia.**

37. Introduction. The object of the experiment is to investigate the relation between the moment of inertia of a system about the axis of the torsion wire by which it is suspended and the periodic time of the torsional vibrations of the body. A convenient system is shown in Fig. 12. The inverted T-piece is formed of two pieces of brass rod of circular section. It is suspended by a torsion wire whose ends are soldered into small brass rods about 0·5 cm in diameter. These rods fit into holes in the rod *E* and the rod *CD*, and are secured by set-screws. A piece of pianoforte steel wire, which has been heated to redness by a flame while

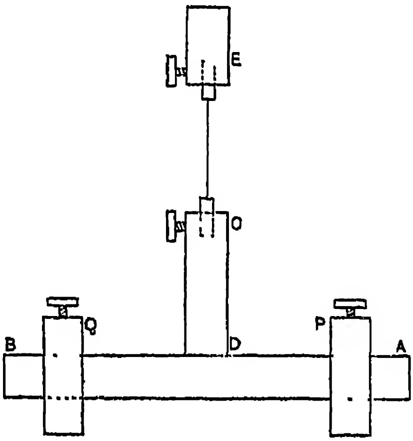

Fig. 12.

under tension, forms a good torsion wire, since it is quite straight and is stiff enough to escape being accidentally bent. The wire used in § 39 was 15 cm. in length and 0·09 cm. in diameter. The bar *AB* carries two short cylinders *P*, *Q*, the axes of these cylinders coinciding with the axis of *AB*; the cylinders are secured to the rod by set-screws.

The rod *E* is held firmly in a suitable unyielding clamp, or in any other way, so that the torsion wire is vertical.

The masses of the two moveable cylinders should be nearly equal, so that the mean mass may be used without appreciable error.

38. Theory of experiment. Let the moment of inertia of the T-piece about the axis of the rod CD be K_0 gm. cm.^{2}, let the masses of the moveable cylinders P, Q be m_1, m_2 gm, and let the moments of inertia of the cylinders P, Q about axes through their centres of gravity at right angles to their axes of figure be K_1 and K_2 gm. cm.^{2}. Let the distances of the centres of gravity of P and Q from the axis of CD be r_1 and r_2 cm. Then, by the theorem of parallel axes*, if the moment of inertia of the whole system about the axis of CD be K gm. cm.^{2}, we have

$$K = K_0 + K_1 + K_2 + m_1 r_1^2 + m_2 r_2^2 \quad \ldots\ldots\ldots\ldots(1)$$

Let $m_1 = m + n$, $m_2 = m - n$, where m is the mean of the two masses. Let $r_1 = \frac{1}{2}d + x$, $r_2 = \frac{1}{2}d - x$, where $d = r_1 + r_2$, the distance between the two centres of gravity. Then

$$m_1 r_1^2 + m_2 r_2^2 = \tfrac{1}{2}md^2 + 2mx^2 + 2nxd.$$

If the two cylinders have been carefully turned and if care is taken to adjust them on the rod AB so that they are as nearly as possible equidistant from CD, both n and x will be so small that we may write

$$m_1 r_1^2 + m_2 r_2^2 = \tfrac{1}{2}md^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(2)$$

and thus (1) becomes

$$K = K_0 + K_1 + K_2 + \tfrac{1}{2}md^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(3)$$

It is thus only necessary to measure the whole distance d or $r_1 + r_2$, i.e. the distance between the centres of gravity of the moveable cylinders, and not the two separate distances r_1 and r_2. Measurements must, of course, be made to enable the two cylinders to be set symmetrically on the rod AB, but no further use is made of those measurements. The distance d is measured *after* the cylinders have been arranged as nearly as possible symmetrically.

If the couple required to twist the system against the torsion of the wire be μ dyne-cm. per radian, we see, by § 33, that, if the periodic time be T secs.,

$$T^2 = (4\pi^2/\mu)\{K_0 + K_1 + K_2 + \tfrac{1}{2}md^2\}. \quad \ldots\ldots\ldots\ldots(4)$$

Hence, if d be varied and the corresponding values of T^2 found experimentally be plotted against d^2, the points ought to lie on a straight line.

* *Experimental Elasticity*, Note IV.

If T and T' be the periodic times corresponding to the distances d and d', we have from (4)

$$\frac{T^2}{T'^2}=\frac{K_0+K_1+K_2+\frac{1}{2}md^2}{K_0+K_1+K_2+\frac{1}{2}md'^2},$$

and thus

$$K_0+K_1+K_2=\frac{m(T'^2d^2-T^2d'^2)}{2(T^2-T'^2)}. \quad \ldots\ldots\ldots\ldots(5)$$

Hence $K_0+K_1+K_2$ can be determined from two observations or from two points on the straight line which lies most evenly among the plotted points when a series of observations is made.

The value of $K_0+K_1+K_2$ given by (5) may be compared with that deduced from the masses and dimensions of the bodies.

If the length of the rod AB be $2l$ cm. and its diameter $2a$ cm., and if its mass be M gm., its moment of inertia about the axis of CD is $M(\frac{1}{3}l^2+\frac{1}{4}a^2)$*. If the diameter of the bar CD be $2c$ cm and if its mass be M' gm, its moment of inertia about its own axis is $\frac{1}{2}M'c^2$, and hence

$$K_0=M(\tfrac{1}{3}l^2+\tfrac{1}{4}a^2)+\tfrac{1}{2}M'c^2.$$

Here M is the mass of AB before any hole is bored in it; M' may be taken as the difference between the whole mass of the T-piece and the mass of AB. These masses should be marked on the rods.

Let the mean length of each moveable cylinder be $2h$ cm and let the external diameter be $2b$ cm., then, since the internal diameter is $2a$, we obtain by the method of § 55

$$K_1+K_2=2m\{\tfrac{1}{3}h^2+\tfrac{1}{4}(b^2+a^2)\}.$$

39. Practical example. The observations may be entered as in the following record of an experiment by Miss B G E. Hooke.

Length of bar $AB=2l=25\ 00$ cm. Hence $l=12\ 5$ cm.
Diameter of $AB=2a=1{\cdot}26$ cm. Hence $a=0\ 63$ cm.
Mass of $AB=M=267$ gm.
Moment of inertia of $AB=M\left(\frac{l^2}{3}+\frac{a^2}{4}\right)=267\left(\frac{12\ 5^2}{3}+\frac{0{\cdot}63^2}{4}\right)=13932$ gm. cm.^{2}.
Diameter of bar $CD=2c=1{\cdot}26$ cm.
Mass of $CD=M'=126$ gm.
Moment of inertia of $CD=\frac{1}{2}M'c^2=\frac{1}{2}\times 126\times 0{\cdot}63^2=25{\cdot}0$ gm. cm.^{2}.
Total moment of inertia of T-piece $=K_0=13957$ gm cm^2.

* *Experimental Elasticity*, Note IV.

Masses of sliding cylinders; $m_1 = 54·82$ gm ; $m_2 = 54·62$ gm.
Mean mass $= m = \frac{1}{2}(m_1 + m_2) = 54·72$ gm.
External diameter of cylinders $= 2b = 3·14$ cm. Hence $b = 1·57$ cm.
Internal diameter of cylinders $= 2a = 1·26$ cm. Hence $a = 0·63$ cm.
Length of cylinders; $2h_1 = 1·02$ cm ; $2h_2 = 1·04$ cm.
Mean length $= 2h = h_1 + h_2 = 1·03$ cm. Hence $h = 0·515$ cm.
Hence for the sliding cylinders

$$K_1 + K_2 = 2m\{\tfrac{1}{3}h^2 + \tfrac{1}{4}(b^2 + a^2)\} = 109·4\{\tfrac{1}{3}\times 0·515^2 + \tfrac{1}{4}(1·57^2 + 0·63^2)\}$$
$$= 87·95 \text{ gm cm.}^2$$

Hence $$K_0 + K_1 + K_2 = 13957 + 88 = 14045 \text{ gm. cm.}^2$$

A series of observations was made in which the distance between the centres of the cylinders P and Q was changed.

Distance d cm.	Number of vibrations	Time secs.	Periodic time (T') secs.	T'^2 secs.^{2}	d^2 cm.^{2}
23·97	70	130·0	1·859	3·456	574·6
	70	130·2			
20·23	80	136·4	1·706	2·910	409·3
	80	136·6			
15·86	90	140·0	1·556	2·421	251·5
	90	140·0			
12·30	100	144·6	1·447	2·094	151·3
	100	144·8			
8·30	100	135·0	1·351	1·825	68·89
	100	135·2			
4·33	100	129·8	1·298	1·685	18·75
	100	129·8			

The observations were then plotted with d^2 as abscissa and T'^2 as ordinate and the points were found to lie very nearly on a straight line. The value of $(K_0 + K_1 + K_2)$ was deduced from the straight line lying most evenly among the points. From this straight line it was found that, when $d^2 = 0$, $T'^2 = 1·625$ secs.^{2}, and that, when $d^2 = 574·0$ cm.^{2}, $T'^2 = 3·45$ secs.^{2}. Hence putting $d' = 0$ in (5) we have

$$K_0 + K_1 + K_2 = \frac{54·72 \times 1·625 \times 574}{2(3·45 - 1·625)} = 13984 \text{ gm. cm.}^2$$

The value found directly was 14045 gm. cm.^{2} and thus the difference between the two values of $(K_0 + K_1 + K_2)$ is less than 1 part in 200.

EXPERIMENT 5. Dynamical determination of the ratio of couple to twist for a torsion wire.

40. Introduction. One of the most important applications of harmonic motion is the determination of the ratio of the couple

to the twist when a system is capable of vibration about a vertical axis. In the present experiment the system is suspended by a torsion wire, and thus the restoring couple is proportional to the angle through which the system is twisted, unless the angle is so large that Hooke's law begins to fail.

If the restoring couple is $\mu\theta$ dyne-cm. when the angle is θ radians and if the moment of inertia of the body about the axis of suspension is K gm. cm², the motion is harmonic and the periodic time T is given (§ 33) by

$$T = 2\pi\sqrt{\frac{K}{\mu}} \text{ seconds. } \qquad \ldots\ldots(1)$$

Hence, if K be calculated from the mass and dimensions of the body and if T be observed, we can find μ from the formula

$$\mu = \frac{4\pi^2 K}{T^2} \text{ dyne-cm. per radian. } \qquad \ldots\ldots(2)$$

41. Experimental details. The two ends of the wire AB (Fig. 13) are soldered into two pieces of rod C, D about $\frac{1}{2}$ cm. in diameter. The rod C is held in a suitable firm clamp so that the wire is vertical and the rod D passes through an inertia bar E, being secured by a set-screw S.

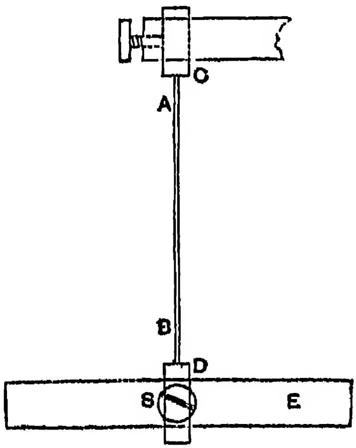

Fig. 13.

The time occupied by a large number of *complete* vibrations is found at least twice, the observation in each case extending over at least two or three minutes—the longer the better, if accuracy is required. The mean time of a complete vibration (T secs.) is then deduced. Unless the time-piece in use be known to be keeping good time, it should be compared* with a good clock to find the necessary correction.

* See § 26.

A stop-watch is generally used in observing the time of vibration. But very good results can be obtained by the method described in §§ 26, 27, using an ordinary watch or clock fitted with a seconds hand.

The accuracy of the work may be tested by using two or more inertia bars and comparing the values of μ found with them. If the rod soldered to the end of the torsion wire be long enough, two or more bars can be clamped to it.

The moment of inertia of each bar is calculated from its mass and dimensions. If the bar be rectangular and if its length be $2L$ cm., its width $2A$ cm., and its mass M gm., and if K be its moment of inertia about an axis through its centre at right angles to $2L$ and $2A$,

$$K = \tfrac{1}{3}M(L^2 + A^2). \qquad \text{.......... .. ..(3)}$$

The fundamental difficulty in every experiment with inertia bars is the impossibility of knowing that the density of the bar is uniform. It is found that there may be sensible differences of density at different points of a brass bar. The formula (3) assumes uniform density, and may be inaccurate if the density is not uniform One way of meeting this difficulty is to use a number of bars and to find the value of μ for a given suspension by aid of each bar in turn. If a number of the bars give closely concordant values for μ, there is great probability that the values of K assumed for those bars are nearly correct.

42. Practical example. The observations may be entered as in the following record of an experiment on a brass wire. Two inertia bars were used.

Mass of bar (1) $= M_1 = 800\,7$ gm.

Length $= 2L_1 = 37{\cdot}88$ cm. Width $= 2A_1 = 1{\cdot}58$ cm.

Hence $K_1 = \tfrac{1}{3}M_1(L_1^2 + A_1^2) = \tfrac{1}{3} \times 800{\cdot}7\,(18{\cdot}94^2 + 0{\cdot}79^2)$

$= 9\,591 \times 10^4$ gm. cm.^{2}.

Mass of bar (2) $= M_2 = 655$ gm.

Length $= 2L_2 = 30\,02$ cm. Width $= 2A_2 = 1{\cdot}60$ cm.

Hence $K_2 = \tfrac{1}{3} \times 655\,(15{\cdot}01^2 + 0{\cdot}80^2) = 4{\cdot}933 \times 10^4$ gm. cm.^{2}.

The following results were obtained, using the two bars separately and together; a stop-watch was used. The moment of inertia of the two bars together is $K_{1+2} = 14{\cdot}524 \times 10^4$ gm. cm.^{2}.

The value of μ was found in each case by the formula

$$\mu = 4\pi^2 K/T^2.$$

Moment of inertia $= K$ gm. cm.^{2}	Number of vibrations	Time secs.	Periodic time $= T$ secs	Ratio of couple to twist $= \mu$ dyne-cm. per radian
9.591×10^4	50	126·0	2·516	5.981×10^5
	50	125·5		
	50	125·8		
4.933×10^4	100	180·1	1·802	5.998×10^5
	100	180·4		
	100	180·1		
14.524×10^4	50	155·1	3·094	5.990×10^5
	50	154·7		
	50	154·4		

EXPERIMENT 6. **Comparison of the moments of inertia of two bodies.**

43. **Theory of experiment.** When a rigid body suspended by a vertical torsion wire vibrates in the periodic time T secs, its moment of inertia (K gm. cm.^{2}) about the axis of suspension is given by

$$K = \frac{\mu T^2}{4\pi^2}, \quad \text{.........................} \quad (1)$$

where the couple required to turn the suspended body through one radian against the torsion of the wire is μ dyne-cm. We can use this expression to compare the values of K for two bodies if we clamp them in turn to the torsion wire and find the corresponding periodic times. Thus, if T_1 and T_2 secs. be the periodic times of the two bodies whose moments of inertia about the axis of suspension are K_1 and K_2 gm. cm.^{2}, we have

$$\frac{K_1}{K_2} = \frac{T_1^2}{T_2^2}. \quad \text{..............................} \quad (2)$$

The operation of timing is carried out exactly as in EXPERIMENT 5. If the value of K_1 be found by calculation from the mass and dimensions of the first body, the moment of inertia of the second body can be deduced by (2).

44. **Practical example.** The observations may be entered as in the following record of an experiment.

Mass of rectangular inertia bar $= M_1 = 800{\cdot}7$ gm.
Length $= 2L_1 = 37{\cdot}88$ cm. Width $= 2A_1 = 1{\cdot}58$ cm.
Hence $K_1 = \frac{1}{3} M_1 (L_1^2 + A_1^2) = 9{\cdot}591 \times 10^4$ gm. cm.^{2}.

The time of 50 vibrations was 126·0, 125 5, 125 8; mean = 125·8 secs

Hence $T_1 = 125\ 8/50 = 2\ 516$ secs

The second body was also a rectangular inertia bar. The time of 100 vibrations was 180 1, 180 4, 180 1, mean = 180·2 secs.

Hence $T_2 = 1\ 802$ secs.

By (2), $K_2 = \frac{K_1 T_2^2}{T_1^2} = \frac{9\ 591 \times 10^4 \times 1{\cdot}802^2}{2\ 516^2} = 4{\cdot}920 \times 10^4$ gm cm.^{2}.

The moment of inertia of the second body was calculated from its mass and dimensions.

Mass of rectangular bar $= M_2 = 655$ gm

Length $= 2L_2 = 30\ 02$ cm Width $= 2A_2 = 1\ 60$ cm.

Hence $K_2 = \frac{1}{3}M(L_2^2 + A_2^2) = 4\ 933 \times 10^4$ gm cm ^{2}

The discrepancy between the two values of K_2 is only about one part in four hundred.

EXPERIMENT 7. **Experiment with a pair of inertia bars.**

45. **Introduction.** Two inertia bars A, B (Fig. 14) are connected together by a wire CD, and the whole system is suspended by a fine line of *plaited* silk (fishing line) two or three metres in length*. The ends of the wire are soldered into short rods about 0 5 cm. in diameter, and these rods are secured in the inertia bars by set-screws. Each of the two rods is furnished with a small hook, by which it can be attached to the silk line.

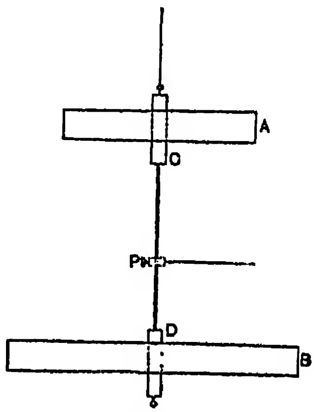

Fig 1

The torsional rigidity of such a silk suspension is so small compared with that of the wire CD that for all practical purposes it may be neglected. We thus obtain a system which is practically free from any applied couple. Its resultant angular momentum is therefore constant. In EXPERIMENT 14 the silk line is replaced by a wire and the investigation then becomes more difficult.

If the two bars are set vibrating in opposite directions, it will be possible, by applying a couple at any point of the wire, to

* It is necessary to use a *plaited* and not twisted cord, since, if twisted cord is used, it exerts a couple upon the suspended system.

adjust the vibrations so that, when one bar is instantaneously at rest at the extremity of its vibration, the other bar is also instantaneously at rest. In this case, the resultant angular momentum is zero, and it will remain zero since the silk suspension exerts no appreciable couple upon the system. Since the two bars move in opposite directions, there must be some point P on the wire CD, where the wire does not rotate. This point can be discovered by attaching to the wire a light pointer (knitting needle) by means of the small clamp shown in Fig. 15. The bars are set into vibration and the pointer is then held fast for a moment and then is released. It is easy to see whether the pointer now vibrates or not, and, if it does vibrate, to decide whether it moves in the same direction as the upper bar or in the same direction as the lower bar.

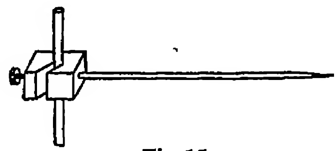

Fig. 15.

46. Theory of experiment. Since the point P is at rest, the bar A vibrates as if it were suspended by the torsion wire CP and the bar B vibrates as if it were suspended by the torsion wire DP. On the assumption that the wire is *uniform*, we can easily calculate the position of P. Let the moments of inertia of the bars A, B about the axis of suspension be K_1, K_2 gm. cm.^{2}. Let the length of the wire CD be l cm., measured between the ends of the two small rods into which the wire is soldered. Let $CP = l_1$ and $DP = l_2$ cm. Let the couple required to twist one end of a piece of the wire one cm. in length through one radian with respect to the other end be η dyne-cm. Then, the couple required to twist one end of CP through one radian with respect to the other end is η/l_1 dyne-cm.

Hence, if the periodic time be T_3 secs., we have (§ 33)

$$T_3 = 2\pi\sqrt{\frac{K_1}{\eta/l_1}} = 2\pi\sqrt{\frac{K_1 l_1}{\eta}}. \qquad \text{.................(1)}$$

The periodic time of the other bar is also T_3, and thus

$$T_3 = 2\pi\sqrt{\frac{K_2 l_2}{\eta}}. \qquad \text{.........................(2)}$$

Hence $$K_1 l_1 = K_2 l_2, \qquad \text{..........................(3)}$$

and thus the point P may be described as being at the "centre of gravity" of the two moments of inertia.

If the system be removed from the silk line and if the bar A be clamped to a firm support so that the wire CD is vertical, the bar B, when set into vibration, will vibrate under the influence of the whole length l of the wire. Hence, if T_2 secs. be its periodic time,

$$T_2 = 2\pi\sqrt{\frac{K_2 l}{\eta}} \qquad \text{.........................(4)}$$

If B be clamped and A be allowed to vibrate, its periodic time will be

$$T_1 = 2\pi\sqrt{\frac{K_1 l}{\eta}} \qquad \text{.........................(5)}$$

From (1) and (5) we have

$$l_1 = lT_3^2/T_1^2$$

and from (2) and (4)

$$l_2 = lT_3^2/T_2^2.$$

Since $l_1 + l_2 = l$, we obtain

$$\frac{1}{T_3^2} = \frac{1}{T_1^2} + \frac{1}{T_2^2}. \qquad \text{.........................(6)}$$

We have obtained (6) on the assumption that the wire is uniform. But its truth does not depend upon the uniformity of the wire, and we shall now give a second proof.

Let the couple required to twist one end of the wire CD through one radian relative to the other end be μ dyne-cm. Then, if the bar A is displaced through an angle θ_1 in one direction and the bar B through θ_2 in the *same* direction, the couple exerted by the wire is $\mu(\theta_1 - \theta_2)$ dyne-cm. The wire acts on A in the direction tending to diminish θ_1 and on B in the direction tending to increase θ_2. Hence the equations of motion of the two bars are

$$K_1\frac{d^2\theta_1}{dt^2} = -\mu(\theta_1 - \theta_2), \qquad K_2\frac{d^2\theta_2}{dt^2} = \mu(\theta_1 - \theta_2). \qquad \text{......(7)}$$

Hence

$$\frac{d^2}{dt^2}(\theta_1 - \theta_2) = -\left(\frac{\mu}{K_1} + \frac{\mu}{K_2}\right)(\theta_1 - \theta_2).$$

The angle $\theta_1 - \theta_2$ between the two bars thus varies harmonically with the time. If T_3 be the periodic time, we have, by § 10,

$$T_3 = 2\pi \sqrt{\frac{1}{\mu(1/K_1 + 1/K_2)}}. \qquad \dots \text{(8)}$$

But
$$T_1 = 2\pi \sqrt{\frac{K_1}{\mu}}, \qquad T_2 = 2\pi \sqrt{\frac{K_2}{\mu}},$$

and hence
$$\frac{1}{T_3^2} = \frac{1}{T_1^2} + \frac{1}{T_2^2}, \qquad \dots \text{(9)}$$

as we found before.

Since $\theta_1 - \theta_2$ varies harmonically, we may put $\theta_1 - \theta_2 = \beta \sin pt$, where $p = 2\pi/T_3$. Then, by (7),

$$K_1 \frac{d^2\theta_1}{dt^2} = -\mu\beta \sin pt.$$

The integral of this equation is

$$\theta_1 = G + Ht + (\mu\beta/K_1p^2) \sin pt = G + Ht + (\beta T_3^2/T_1^2) \sin pt,$$

where G and H are constants. Thus the motion of the bar A is the resultant of a uniform rotation with angular velocity H and a harmonic motion. Since $\theta_1 - \theta_2 = \beta \sin pt$, we find by (9) that the position of the bar B is given by

$$\theta_2 = \theta_1 - \beta \sin pt = G + Ht - (\beta T_3^2/T_2^2) \sin pt,$$

so that the motion of the bar B is of the same type as that of the bar A.

47. Experimental details. The two bars are attached to the torsion wire, and each is clamped to a steady support while the periodic time of the other is being found. The ratio of the squares of the periodic times, T_1^2/T_2^2, is compared with the ratio of the moments of inertia of the bars (§ 43), if these are of such forms that their moments of inertia are easily calculated. With careful work the two ratios will agree closely. The system is now suspended by the silk line and the bars are set vibrating in opposite directions. The pointer is adjusted on the wire so that it will remain at rest if it is released after having been brought to rest while the bars are vibrating. The observer is aided in this adjustment by noticing whether the pointer moves in the same direction as the upper bar or in the opposite direction. A good deal of care is needed to get a close adjustment.

The periodic time, T_3, of the two bars is now observed. If th pointer has a very slight angular velocity, this may be checke from time to time so as to keep the pointer very approximately i a constant direction

There is a point on the wire such that, if the pointer clamped there, a couple applied to the clamp will cause the tw bars, supposed initially at rest, to move round as if they wer rigidly connected. Let the distances of this point from the en C, D of the wire be h_1, h_2 cm. Then, if the bars and the pointe are initially in one plane and θ_1, θ_2, ϕ are the angles betwee them and that plane at any subsequent time, we have, when th wire is uniform,

$$\frac{\eta}{h_1}(\phi-\theta_1)=K_1\frac{d^2\theta_1}{dt^2}, \qquad \frac{\eta}{h_2}(\phi-\theta_2)=K_2\frac{d^2\theta_2}{dt^2}.$$

If $\theta_1=\theta_2$ always, then $d^2\theta_1/dt^2=d^2\theta_2/dt^2$, and hence $h_1K_1=h_2K$ The position of the pointer is therefore the same as that foun in § 46.

48. Practical example. The observations may be entered as i the following record of an experiment by G. F. C Searle

A brass wire was used; it was 0·15 cm. in diameter and was 60 cm. i length between the two rods into which it was soldered. The inertia ba were of rectangular form.

Mass of bar $A=M_1=826$ gm.
Length$=2L_1=37\ 88$ cm. Width$=2A_1=1{\cdot}60$ cm.
Hence $K_1=\frac{1}{3}M_1(L_1^2+A_1^2)=9{\cdot}894\times10^4$ gm. cm.^{2}.
Mass of bar $B=M_2=655$ gm.
Length$=2L_2=30{\cdot}02$ cm Width$=2A_2=1{\cdot}60$ cm.
Hence $K_2=\frac{1}{3}M_2(L_2^2+A_2^2)=4{\cdot}933\times10^4$ gm. cm.^{2}.

Then $$\frac{K_1}{K_2}=\frac{9{\cdot}894\times10^4}{4{\cdot}933\times10^4}=2{\cdot}006.$$

The time of 100 vibrations of the bar A, when B was fixed, was 366·4, 366· 367·2; mean 366·5 secs.

Hence $T_1=3{\cdot}665$ secs

The time of 100 vibrations of the bar B, when A was fixed, was 259· 258 7, 259 0; mean 259·0 secs.

Hence $T_2=2{\cdot}590$ secs.

Then, by § 43, $$\frac{K_1}{K_2}=\frac{T_1^2}{T_2^2}=\frac{3{\cdot}665^2}{2{\cdot}590^2}=2{\cdot}002.$$

This result agrees closely with that found from the dimensions of the ba

The time of 100 vibrations of the two bars when the system was suspended by the silk line was 211 2, 211 8, 211 4; mean 211·5 secs.

Hence $T_3 = 2\ 115$ secs.

Then $$1/T_3^2 = 1/2{\cdot}115^2 = 0{\cdot}2236,$$

and $$1/T_1^2 + 1/T_2^2 = 1/3\ 665^2 + 1/2\ 590^2 = 0{\cdot}0744 + 0{\cdot}1491 = 0\ 2235$$

Hence the formula (9) is very closely verified.

It was found that when the pointer was fixed to the wire so that $CP = 20$ cm., $DP = 40$ cm (Fig. 14) it remained stationary when the bars vibrated.

EXPERIMENT 8. Determination of the moment of inertia of a rigid pendulum.

49. Theory of rigid pendulum. Consider a rigid body of mass M gm. which can turn about a fixed horizontal axis, this axis cuts the plane of the paper in Fig. 16 at right angles. Let the body be displaced so that the perpendicular GO from the centre of gravity G upon the horizontal axis makes an angle θ radians with the vertical OZ. Let OG be h cm. Then the distance GN between the two verticals through O and G is $h \sin\theta$ cm., and, therefore, the moment about O of the weight of the pendulum is $Mgh \sin\theta$ dyne-cm., since the action of gravity on the pendulum is equivalent to a single vertical force Mg dynes acting downwards at G. This moment always acts so as to tend to make OG coincide with OZ. The reaction of the axis or knife-edge passes through the axis of rotation, and hence has no moment about it.

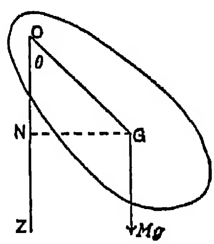

Fig. 16.

If the moment of inertia of the body about the horizontal axis be K gm. cm.^{2}, and if the angular acceleration of the line OG be α radians per sec. per sec. in the positive direction of θ, we have*

$$K\alpha = -Mgh \sin\theta,$$

or $$\alpha = -\frac{Mgh}{K} \sin\theta. \quad \text{....................(1)}$$

* *Experimental Elasticity*, Note III.

This angular acceleration is exactly the same as that of a simple pendulum of length l, where

$$l = \frac{K}{Mh},$$

and the rigid body will, therefore, vibrate in exactly the same time as the simple pendulum if the arcs of vibration be equal, and this will be the case whether the arc is large or small. If the periodic time for an infinitesimal arc be T secs., we have, as in § 24,

$$T = 2\pi\sqrt{\frac{l}{g}} = 2\pi\sqrt{\frac{K}{Mgh}}. \quad \text{..... ....(2)}$$

The length l is called the length of the simple equivalent pendulum.

Equation (1) may also be obtained by the principle of energy. Let β be the maximum value of θ; then when $\theta = \beta$, the body is instantaneously at rest, and has no kinetic energy. When the body has moved so that $GOZ = \theta$, the centre of gravity has descended through a distance $h(\cos\theta - \cos\beta)$ from the position it had when $\theta = \beta$. The work done by gravity is therefore $Mgh(\cos\theta - \cos\beta)$ ergs, and this must be equal to the kinetic energy of the body. Since the angular velocity of the body about the axis is $d\theta/dt$ radians per sec., the kinetic energy is $\frac{1}{2}K(d\theta/dt)^2$ ergs. Hence

$$\tfrac{1}{2}K\left(\frac{d\theta}{dt}\right)^2 = Mgh(\cos\theta - \cos\beta).$$

Differentiating with respect to the time, we have

$$K\frac{d\theta}{dt}\cdot\frac{d^2\theta}{dt^2} = -Mgh\sin\theta\,.\,\frac{d\theta}{dt},$$

or

$$K\frac{d^2\theta}{dt^2} = -Mgh\sin\theta. \quad \text{...................(3)}$$

Since $d^2\theta/dt^2$ is the angular acceleration, (3) is identical with (1).

We could use equation (2) in determining g if we knew T, M, K and h, but, though T, M and h are easily found, it is impossible to calculate K with any accuracy from the form of the body, since this would require a knowledge of the density of every part of the body, in addition to an accurate knowledge of the form of the body. For this reason equation (2) is useless as a means of

finding g In accurate determinations of g this difficulty is met by the use of Kater's pendulum, which is caused to vibrate about each of two parallel knife-edges in turn, the moment of inertia of the pendulum about an axis through its centre of gravity parallel to the knife-edges is eliminated by this procedure.

A method in which the moment of inertia of the pendulum is found by comparison with that of a bar of simple geometrical form, assumed to be of uniform density, is given in EXPERIMENT 10.

We shall use equation (2) to enable us to determine the value of K for the rigid body. Thus

$$K = \frac{T^2 Mgh}{4\pi^2}. \qquad \ldots\ldots\ldots\ldots(4)$$

If we can balance the body on a knife-edge or fulcrum we can measure h directly, and we can then find the product Mh. But it may happen, on account of the form of the body, that it is difficult to do this, and in that case we must use a method which enables us to find Mh without measuring h directly. The two methods are illustrated below.

50. Determination of periodic time. Before proceeding to obtain the periodic time, the student should ascertain whether the bob of the pendulum (if moveable) is in the standard position. Unless this is done, the demonstrators will not be able to judge of the accuracy of the results obtained. The periodic time T is determined either by finding by a stop-watch the time of 100 or more complete vibrations or by the method used in § 26 in timing the simple pendulum. If a stop-watch is used, *at least* two sets of observations should be made. If necessary, a correction must be made for the gaining or losing of the time-piece. The amplitude of vibration should not exceed a few degrees. (See Note II.)

51. Determination of *Mh*. Method I. When the body is of suitable shape, it can be balanced on a fulcrum so that the centre of gravity G is vertically below or vertically above the fulcrum, in the latter case the equilibrium is unstable. If the body is so balanced that the line OG (Fig. 18), which was vertical when the body was hanging undisturbed from the knife-

edge O, is now horizontal, then the distance between the vertical lines through O and through the fulcrum is the length h.

The mass, M gm., of the body is found by the ordinary process of weighing.

Fig. 18 shows one of the rigid pendulums used at the Cavendish Laboratory for this experiment. A ring carrying a knife-edge O is attached to a steel rod, and a heavy brass cylinder can be clamped to the rod at any point. The knife-edge rests in a V-groove in a support, as in Fig. 17.

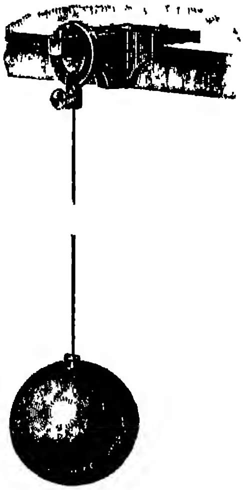

Fig. 17.

The determination of h may be more conveniently carried out in the following manner. A fine groove E (Fig. 18) is turned in the rod within a few millimetres of the centre of gravity of the pendulum when the bob is in the standard position, and the distance of the centre of the groove from O the knife-edge of the pendulum is accurately determined. The pendulum is then placed with the groove E resting on the fulcrum F and a small mass of m gm. is suspended from the pendulum, as in Fig 18, and is adjusted so that

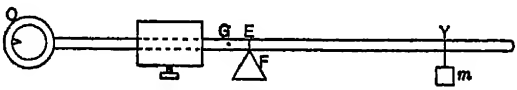

Fig 18.

the pendulum is in (unstable) equilibrium. If Y be the point of suspension of m, and if y cm. be the horizontal distance of Y from the groove E, then the horizontal distance, x cm., of the centre of gravity from F is given by

$$x = \frac{my}{M} \quad \text{.................... (5)}$$

When Y and O lie on *opposite* sides of the fulcrum F, the centre of gravity G lies *between* O and E and

$$h = OG = OE - x. \quad \text{.................. (6)}$$

52. Determination of Mh. Method II. In this method the value of Mh is deduced from the deflexion of the pendulum when a horizontal force is applied to it by a thread which passes over a pulley and supports a weight. The general arrangement of the apparatus is shown in Fig 19. The thread from the ball-bearing pulley L is attached to the pendulum at P, and the point of attachment to the pendulum is adjusted *for each value* of the deflecting mass so that the thread PL is horizontal. Let the direction of the thread cut the vertical OR from O in Q. The adjustment of the thread may be made by aid of a straight edge placed close to the thread PL; the straight edge is set horizontal by a level. By viewing the thread and the straight edge together, it is easy to detect very small deviations of the thread from the horizontal. The vertical planes through the thread and the straight edge should be approximately parallel.

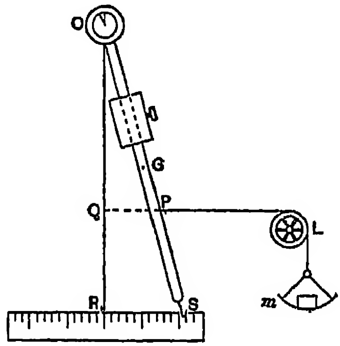

Fig. 19.

The lower end of the pendulum carries a pointer formed of a short piece of knitting needle soldered into a hole in the rod. We shall suppose that, when the pendulum is hanging undeflected, the tip S of the pointer is vertically below O. Then OGS is a straight line. A scale, set horizontal by aid of a level, is arranged so that the horizontal displacement of the tip S can be measured.

The distance OQ is found in the following manner. The reading of the pointer S is taken while the pendulum is undeflected. The pendulum is then *held* in this position while the thread is adjusted on the pendulum so that the thread PL is horizontal, as judged by the straight edge. The distance of the thread from O is ascertained by measuring the distance of the thread from O directly or, more conveniently, from a mark on the pendulum whose distance from O is known.

Let the distance from the knife-edge O to the tip S of the pointer be l cm, let OQ be d cm., and let the angle SOR be θ radians, when the mass supported by the thread is m gm.

Let the horizontal displacement of S from its position when the pendulum is undeflected be x cm. Then, taking moments about O, we have

$$Mgh \sin \theta = mgd. \quad \ldots\ldots\ldots\ldots\ldots (7)$$

But
$$\sin \theta = x/l,$$

and hence
$$Mh = \frac{ld}{x/m}. \quad \ldots\ldots\ldots\ldots\ldots (8)$$

A series of observations is made in which m is varied. The corresponding values of x are found and the mean value of x/m is calculated. It is convenient to adjust the scale-pan, in which the masses are placed, to have some definite mass—say 10 grammes. During the observations, care must be taken not to lean upon the supports of pendulum or pulley or scale.

53. Practical example. The results may be entered as in the following record of an experiment by G. F. C. Searle.

Determination of Mh. Method I.

Mass of pendulum $= M = 1761$ gm.

Distance of groove from knife-edge $= 41{\cdot}37$ cm.

Mass of rider $= 20$ gm.

Distance of rider from fulcrum $= 32$ cm (on opposite side to knife-edge).

Hence $\quad h = 41{\cdot}37 - 20 \times 32/1761 = 41{\cdot}37 - 0{\cdot}36 = 41{\cdot}01$ cm.

Hence $\quad Mh = 1761 \times 41{\cdot}01 = 7{\cdot}222 \times 10^4$ gm. cm.

Determination of Mh. Method II.

Length of pendulum from knife-edge to tip of pointer $= l = 115{\cdot}0$ cm.

Distance of the thread from the knife-edge when the pendulum is vertical $= d = 49{\cdot}82$ cm.

Load (including pan) m gm.	Reading of pointer, cm	Deflexion x cm.	x/m cm. gm $^{-1}$
0	12·24		
20	13·81	1·57	0·0785
40	15·40	3·16	0·0790
60	16·99	4·75	0·0792
80	18·57	6·33	0·0791
100	20·15	7·91	0·0791
120	21·71	9·47	0·0789
140	23·31	11·07	0·0791

Mean value of $x/m = 0{\cdot}0790$ cm. gm.$^{-1}$.

Hence, by (8), $Mh = \dfrac{ld}{x/m} = \dfrac{115\cdot 0 \times 49\cdot 82}{0\cdot 0790} = 7\cdot 252 \times 10^4$ gm cm.

Time of 100 complete vibrations 147·8, 147·3, 147·2, 147·6, mean 147·5 secs.

Periodic time $= T = 1\cdot 475$ secs

By (4),
$$K = \frac{T^2 Mhg}{4\pi^2}$$

By Method I, $K = \dfrac{1\cdot 475^2 \times 7\cdot 222 \times 10^4 \times 981}{4\pi^2} = 3\cdot 904 \times 10^6$ gm cm.^{2}.

By Method II, $K = \dfrac{1\cdot 475^2 \times 7\cdot 252 \times 10^4 \times 981}{4\pi^2} = 3\cdot 921 \times 10^6$ gm cm.^{2}.

EXPERIMENT 9. **Experiment on a pendulum with variable moment of inertia.**

54. Introduction. The properties of moments of inertia are well illustrated in the following experiment on a rigid pendulum. A pendulum rod, as illustrated in Fig. 18, is used and two sets of observations are made. In one set the cylindrical bob is attached to the rod in a position defined by two small grooves turned in the rod, one at each end of the bob In the second set of observations the bob is removed from the rod.

Let the moment of inertia of the unloaded pendulum about its knife-edge be K_0 gm. cm.^{2} and let that of the loaded pendulum be K_1 gm. cm.^{2}. Let the mass of the unloaded pendulum be M_0 gm. and let the mass of the bob be M gm., let the distance of the centre of gravity of the bob from the knife-edge of the pendulum be l cm. and let the moment of inertia of the bob about an axis through its centre of gravity parallel to the knife-edge of the pendulum be Mk^2. Then, if the moment of inertia of the bob about the knife-edge be K_2, we have, by the theorem of parallel axes*,

$$K_2 = Mk^2 + Ml^2. \quad \ldots\ldots\ldots\ldots\ldots\ldots(1)$$

The experiment is designed to test the agreement between K_1 and $K_0 + K_2$.

55. Experimental details. The periodic time, T_0 secs., of the unloaded pendulum is found as in § 50. The mass, M_0 gm., is found by weighing, and the distance, h_0 cm., of the centre of gravity from the knife-edge of the pendulum is found as in § 51 by

* *Experimental Elasticity*, Note IV.

balancing the pendulum on a fulcrum. The moment of inertia of the unloaded pendulum about its knife-edge is then calculated by the formula

$$K_0 = \frac{T_0^2 M_0 g h_0}{4\pi^2}. \qquad (2)$$

The observations are repeated on the loaded pendulum. If the periodic time be T_1 secs, and the distance of the centre of gravity of the loaded pendulum from its knife-edge be h_1 cm., the moment of inertia of the loaded pendulum about its knife-edge is given by

$$K_1 = \frac{T_1^2 (M_0 + M) g h_1}{4\pi^2}, \qquad (3)$$

since the mass of the loaded pendulum is $M_0 + M$ gm.

The bob, which is in the form of a cylinder whose axis coincides with that of the pendulum rod, is now measured and its moment of inertia is calculated. Let its length be $2A$ cm., its external diameter $2R$ cm. and its internal diameter $2r$ cm; the latter may be taken as equal to the diameter of the pendulum rod. If the density of the metal be ρ gm. per c.c., we have

$$\rho = \frac{\text{mass}}{\text{volume}} = \frac{M}{2\pi A (R^2 - r^2)}. \qquad (4)$$

The mass of a solid cylinder of length $2A$ and diameter $2R$ is $2\pi A R^2 \rho$ and its moment of inertia about an axis through its centre of gravity at right angles to its axis of figure is*

$$2\pi A R^2 \rho \left(\tfrac{1}{3}A^2 + \tfrac{1}{4}R^2\right). \qquad (5)$$

The moment of inertia of a solid cylinder of length $2A$ and diameter $2r$ about a similar axis is

$$2\pi A r^2 \rho \left(\tfrac{1}{3}A^2 + \tfrac{1}{4}r^2\right). \qquad (6)$$

The moment of inertia, Mk^2, of the bob about the same axis is the difference between (5) and (6). Using the value found for ρ, we have

$$\begin{aligned} Mk^2 &= 2\pi A \rho \left\{\tfrac{1}{3}A^2 (R^2 - r^2) + \tfrac{1}{4}(R^4 - r^4)\right\} \\ &= M\left(\tfrac{1}{3}A^2 + \tfrac{1}{4}R^2 + \tfrac{1}{4}r^2\right). \qquad (7) \end{aligned}$$

The distance l cm. from the knife-edge of the pendulum to the centre of gravity of the bob is now found. This is equal to the

* *Experimental Elasticity*, Note IV.

distance from the knife-edge to the top of the bob *plus* half the length of the bob. If the workmanship is good, the set-screw will be placed so nearly symmetrically that no appreciable error will result from the assumption that the centre of gravity of the bob, including the screw, lies on a plane midway between the plane ends of the bob. Further, Mk^2 is so small compared with Ml^2 that any error in Mk^2 due to neglecting the screw is quite inappreciable. It is to be clearly understood that M is the *whole* mass of the bob, including the screw.

The value of K_2 is now calculated. Thus, by (1) and (7),

$$K_2 = M(l^2 + \tfrac{1}{3}A^2 + \tfrac{1}{4}R^2 + \tfrac{1}{4}r^2). \qquad (8)$$

The value of $K_0 + K_2$ is then compared with that of K_1.

Since (2) and (3) involve periodic times while (8) depends only on the measurement of a mass and of four lengths, the success of the experiment will depend on the use of a time-piece which is keeping good time. If necessary the time-piece should be compared with a good clock.

56. Practical example. The observations may be entered as in the following record of an experiment by G. F. C. Searle and G. Stead.

Mass of unloaded pendulum $= M_0 = 653$ gm

Distance of groove from knife-edge of pendulum $= 54{\cdot}41$ cm.

Mass of rider $= 10$ gm.

Distance of rider from fulcrum $= 7{\cdot}85$ cm (on opposite side to knife-edge).

Hence $h_0 = 54{\cdot}41 - 10 \times 7{\cdot}85/653 = 54{\cdot}41 - 0{\cdot}12 = 54{\cdot}29$ cm.

Time of 100 complete vibrations 174·2, 174·2, 174·6, mean 174·3 secs

Periodic time $= T_0 = 1{\cdot}743$ secs

Hence, by (2), we find for moment of inertia of unloaded pendulum

$$K_0 = \frac{T_0^2 M_0 g h_0}{4\pi^2} = \frac{1{\cdot}743^2 \times 653 \times 981 \times 54{\cdot}29}{4\pi^2} = 2{\cdot}676 \times 10^6 \text{ gm. cm}^2.$$

Mass of loaded pendulum $= M_0 + M = 653 + 1108 = 1761$ gm.

Distance of groove from knife-edge of pendulum $= 41{\cdot}37$ cm.

Mass of rider $= 20$ gm.

Distance of rider from fulcrum $= 34{\cdot}5$ cm. (on opposite side to knife-edge).

Hence $h_1 = 41{\cdot}37 - 20 \times 34{\cdot}5/1761 = 41{\cdot}37 - 0{\cdot}39 = 40{\cdot}98$ cm.

Time of 100 complete vibrations 147·8, 147·4, 147·6; mean 147·6 secs

Periodic time $= T_1 = 1{\cdot}476$ secs.

Hence, by (3), we find for moment of inertia of loaded pendulum

$$K_1 = \frac{T_1^2 (M_0 + M) g h_1}{4\pi^2} = \frac{1{\cdot}476^2 \times 1761 \times 981 \times 40{\cdot}98}{4\pi^2} = 3{\cdot}907 \times 10^6 \text{ gm. cm}^2.$$

Mass of bob $= M = 1108$ gm. Length $= 2A = 7{\cdot}04$ cm

External diameter $= 2R = 5{\cdot}00$ cm. Internal diameter $= 2r = 0{\cdot}96$ cm

Hence $Mk^2 = M(\frac{1}{3}A^2 + \frac{1}{4}R^2 + \frac{1}{4}r^2) = 1108 \times 5{\cdot}75 = 0{\cdot}00637 \times 10^6$ gm cm.^{2}.

Distance of knife-edge from top of bob $= 29{\cdot}64$ cm.

Hence $l = 29{\cdot}64 + A = 33{\cdot}16$ cm

And $Ml^2 = 1108 \times 33{\cdot}16^2 = 1{\cdot}2183 \times 10^6$ gm cm.^{2}.

Then $K_2 = Mk^2 + Ml^2 = 0{\cdot}00637 \times 10^6 + 1{\cdot}2183 \times 10^6 = 1{\cdot}225 \times 10^6$ gm. cm.^{2}.

Hence $K_0 + K_2 = 2{\cdot}676 \times 10^6 + 1{\cdot}225 \times 10^6 = 3{\cdot}901 \times 10^6$ gm. cm.^{2}.

From the vibrations of the loaded pendulum we found

$$K_1 = 3{\cdot}907 \times 10^6 \text{ gm. cm.}^2.$$

Thus K_1 differs from $K_0 + K_2$ by only 0·15 per cent

EXPERIMENT 10. **Determination of "g" by a rigid pendulum.**

57. **Introduction.** In § 49 it is shown that the periodic time of a rigid pendulum is given by

$$T = 2\pi\sqrt{\frac{K}{Mgh}}, \quad \text{.....................(1)}$$

where K is the moment of inertia of the pendulum about its axis of suspension, h is the distance of the centre of gravity from that axis and M is the mass of the pendulum. It was explained that this equation cannot be used for finding g, since the value of K cannot be satisfactorily found from the mass and dimensions of the pendulum. In the present experiment, the value of K_0, the moment of inertia of the pendulum about an axis through its centre of gravity parallel to the knife-edge, is determined by comparison with the moment of inertia of an inertia bar by the method of torsional vibrations. The value of K, the quantity occurring in (1), is then found by the theorem of parallel axes*. Thus:

$$K = K_0 + Mh^2. \quad \text{......................(2)}$$

58. **Experimental details.** The periodic time (T secs.), of the pendulum, when vibrating about its knife-edge through a small arc, is first observed. At least two independent observations of the time should be made, unless the method of § 26 is used. This time must be accurately determined if an accurate value of g is desired. The mass, M gm., is found by use of a

* *Experimental Elasticity*, Note IV.

balance. The distance, h cm., of the centre of gravity from the knife-edge is found, just as in § 51, by balancing the pendulum on a fulcrum.

When K_0 is to be determined, the pendulum is suspended from a torsion wire in the manner shown in Fig. 20*. The rod of the pendulum rests in a metal stirrup (Fig. 21) and is secured by a set-screw. A short rod soldered to the end of the torsion wire is also secured to the stirrup by a set-screw. If the pendulum is adjusted in the stirrup so that the pendulum rod is horizontal, the centre of gravity of the pendulum will lie on the axis of the torsion wire. The periodic time of the torsional vibrations is now observed at least twice; let it be T_0 secs. An inertia bar of moment of inertia K_1 gm. cm.^{2} is then attached to the torsion wire in place

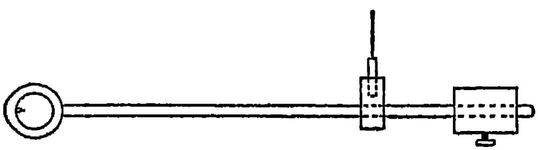

Fig. 20.

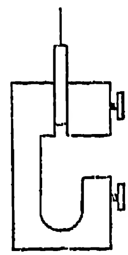

Fig. 21.

of the stirrup and pendulum, and the periodic time, T_1 secs., is observed. Then we have, by § 43, since the moment of inertia of the stirrup is entirely negligible in comparison with that of the pendulum,

$$K_0 = \frac{K_1 T_0^2}{T_1^2} \text{ gm. cm.}^2. \quad \text{.....................(3)}$$

The value of K is then found by (2) and then the gravitational acceleration is found from the formula

$$g = \frac{4\pi^2 K}{T^2 Mh}. \quad \text{...........................(4)}$$

For accuracy, it is obviously desirable that K_0 should be small compared with Mh^2, since, in that case, any uncertainties arising

* In strictness the plane of the ring carrying the knife-edge should be horizontal and not vertical, as shown in Fig. 20, since the two axes (torsion wire and knife-edge) about which the pendulum vibrates are supposed to be parallel.

from the inertia bar or from the effects of imperfect elasticity in the torsion wire will have little influence on the value obtained for g. This result will be secured if the pendulum bob is heavy and the rod is light. The rod might well be in the form of a steel tube.

59. Practical example. The results may be entered as in the following record of an experiment by Miss B. G. E. Hooke and Miss V. Cockburn.

Mass of pendulum $= M = 1761$ gm.

Distance of groove from knife-edge of pendulum $= 88{\cdot}95$ cm.

Mass of rider $= m = 12{\cdot}6$ gm.

Distance of rider from fulcrum for equilibrium $= y = 66$ cm.

Hence, (see § 51), $x = 12{\cdot}6 \times 66/1761 = 0{\cdot}47$ cm.

Hence $h = 88{\cdot}95 + 0{\cdot}47 = 89{\cdot}42$ cm.

Thus $\quad Mh^2 = 1761 \times 89{\cdot}42^2 = 14{\cdot}08 \times 10^6$ gm. cm.^{2}.

Time of 100 complete vibrations of pendulum about knife-edge 202·8, 202·6; mean 202·7 secs.

Periodic time $= T = 2{\cdot}027$ secs.

Time of 10 complete torsional vibrations of pendulum when suspended by torsion wire 237·6, 238·0, 237·4; mean 237·7 secs.

Periodic time $= T_0 = 23{\cdot}77$ secs.

Mass of inertia bar $= M_1 = 1600{\cdot}6$ gm.

Length of bar $= 2L = 37{\cdot}95$ cm.

Width of bar $= 2A = 1{\cdot}60$ cm.

Moment of inertia of bar $= K_1 = \frac{1}{3}M_1(L^2 + A^2) = 1{\cdot}924 \times 10^5$ gm. cm.^{2}.

Time of 50 complete torsional vibrations of inertia bar 365, 363; mean 364 secs.

Periodic time $= T_1 = 7{\cdot}280$ secs.

Moment of inertia of pendulum about axis through centre of gravity $= K_0$.

By (3) $\quad K_0 = \dfrac{T_0^2 K_1}{T_1^2} = \dfrac{23{\cdot}77^2 \times 1{\cdot}924 \times 10^5}{7{\cdot}280^2} = 2{\cdot}052 \times 10^6$ gm. cm.^{2}.

Moment of inertia of pendulum about knife-edge $= K$.

$$K = K_0 + Mh^2 = 2{\cdot}052 \times 10^6 + 14{\cdot}08 \times 10^6 = 1{\cdot}613 \times 10^7 \text{ gm. cm.}^2.$$

Then, by (4)

$$g = \frac{4\pi^2 K}{T^2 Mh} = \frac{4\pi^2 \times 1{\cdot}613 \times 10^7}{2{\cdot}027^2 \times 1761 \times 89{\cdot}42} = 984{\cdot}2 \text{ cm. sec.}^{-2}.$$

EXPERIMENT 11. Pendulum on a yielding support.

60. Introduction. When a simple pendulum is hung from a yielding support, the periodic time differs from that which the pendulum has when the support is fixed. We shall investigate

e case in which the support can move horizontally but not rtically, and we shall suppose that the inertia of e support is negligible.

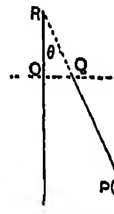

Fig. 22.

Let the point of support Q (Fig. 22) move along e horizontal straight line OQ, and let the mass of e bob be M grammes. Let the force required to splace the point of support through one cm. from its position of equilibrium, be μ dynes. Since the ertia of the support is negligible, the same force ll be required whether Q is at rest or has an acleration.

When the arc of vibration is very small, the tension of the ing QP will differ very little from Mg, the tension it has when e bob hangs at rest.

When the bob swings, the horizontal component of the tension equal to $\mu . OQ$. Hence, if θ be the inclination of the string to e vertical,

$$Mg \sin \theta = \mu . OQ,$$

since θ is small,
$$Mg\theta = \mu . OQ. \quad \text{.......................(1)}$$

it, if PQ intersect the vertical through O in R,

$$RO = OQ/\theta = Mg/\mu,$$

$$\mu . RO = Mg, \quad \text{................................(2)}$$

d thus R is a fixed point.

But $\mu . RO$ is the force required to displace Q through rizontal distance RO and this, by the last equation, we see is ual to Mg. Hence RO is equal to the distance through which the ight of the bob, applied horizontally, would deflect the support.

Since R is a fixed point, the pendulum has the same motion if its string were of length PR and it hung from a fixed support. ence, if the pendulum be hung (1) from a fixed support, when its riodic time is T_1, and (2) from a yielding support, when its periodic ne is T_2, the excess of the second over the first equivalent length equal to the distance through which the weight of the bob, if plied horizontally, would deflect the support.

In an actual experiment, the support will yield vertically as ll as horizontally. But for very small oscillations the vertical ce acting on the support will be very nearly constant and thus

Q will move very nearly in a horizontal line. Hence the actual case will give the same result as the ideal one just investigated.

61. Experimental details. A wooden rod AB (Fig. 23) is clamped at one end to a shelf or other firm structure. The end A is furnished with a V-groove in which rests a knife-edge attached to the ring C; two screws, each with a V-groove cut in its head, form a satisfactory arrangement. The pendulum bob P is hung from the ring by a wire. The ring carries a light cardboard scale D. A straight line is ruled along the scale, and this line is in the same vertical straight line as the wire of the pendulum when the pendulum hangs at rest.

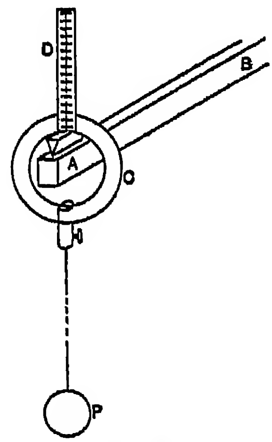

Fig. 23.

The depth of the rod should be considerably larger than the width, so that the vertical yielding of the rod may be small compared with its horizontal yielding.

The pendulum is first hung from a fixed support, as in Fig. 24, and the time occupied by 100 vibrations is observed at least three times. The periodic time, T_1 secs., is calculated and the equivalent length is found from the formula

$$l_1 = \frac{gT_1^2}{4\pi^2} \text{ cm.} \quad \ldots\ldots\ldots\ldots(3)$$

The pendulum is then hung from the yielding support and the periodic time, T_2 secs., is deduced from at least three sets of observations of 100 vibrations. The equivalent length l_2 is found from the formula

$$l_2 = \frac{gT_2^2}{4\pi^2} \text{ cm.} \quad \ldots\ldots\ldots\ldots(4)$$

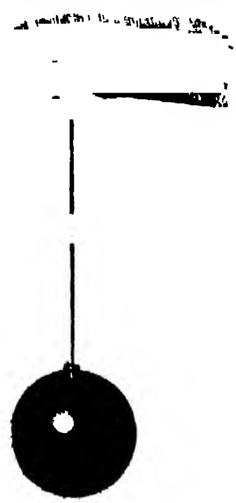
Fig. 24.

The value of $l_2 - l_1$ is then found from (3) and (4).

Since absolute and not merely relative

values of T_1 and T_2 are required, the watch must be one keeping correct time.

A telescope with a vertical cross-wire is now directed towards the scale D and the straight line ruled on the scale is observed. It will be found that, when the pendulum vibrates, there is one point on the straight line which remains at rest, and the scale reading of this point is read. The scale reading, on the same scale, of the knife-edge is also found. The difference of scale readings gives the distance OR in Fig. 22.

The effect of a horizontal force applied to the point of support is now examined.

A ball-bearing pulley K (Fig. 25) is arranged so that the string KA is horizontal and also perpendicular to the length of the rod. The string supports a light pan F. The bob is allowed to hang in a vessel V containing water, for the purpose of damping the vibrations. A horizontal scale S is arranged for finding the deflexion of the support due to a load in the pan. Care must be taken that the bob does not touch the sides of the vessel.

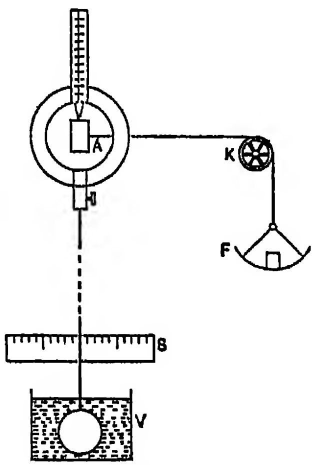

Fig. 25.

A series of observations is made in which the load N is varied, the deflexion x being noted for each load. Two sets of these observations are made with the pulley on opposite sides of the rod. The zero reading in each case may be taken when the pan alone is supported by the string. From these observations the mean value of μ is found from the equation

$$\mu = \frac{N}{x} g.$$

The distance RO is now found from the formula (2)

$$RO = \frac{Mg}{\mu} = M \frac{x}{N}. \quad \text{.....................(5)}$$

The values of RO found by the three methods are then compared.

62. Practical example. The observations may be entered as in the following record of an experiment by Mr D H. Peacock.

The length of the wooden rod projecting from the clamp was 109 cm. The width of the rod was 1 cm. and the depth was 3 cm.

Mass of bob $= M = 1455$ gm.

Pendulum on fixed support · 100 vibrations in 276·5, 276 2, 276·1 secs.

Mean value of $T_1 = 2\ 763$ secs.

Pendulum on yielding support: 100 vibrations in 289 3, 289·3, 289·0 secs.

Mean value of $T_2 = 2\ 892$ secs.

Hence, by (3) and (4),

$$l_1 = \frac{T_1^2 g}{4\pi^2} = \frac{2\ 763^2 \times 981}{4\pi^2} = 189{\cdot}70 \text{ cm.}$$

$$l_2 = \frac{T_2^2 g}{4\pi^2} = \frac{2{\cdot}892^2 \times 981}{4\pi^2} = 207{\cdot}83 \text{ cm.}$$

Hence $$l_2 - l_1 = 18{\cdot}13 \text{ cm.}$$

The deflexion due to the horizontal force was then observed. The following results were obtained

Deflexion to left			Deflexion to right		
Load gm.	Reading cm.	Deflexion / Load	Load gm.	Reading cm.	Deflexion / Load
0	13 80		0	13 15	
100	15·05	·01250	100	11 85	·01300
200	16 33	01265	200	10 52	01315
300	17·60	01267	300	9·25	·01300
400	18·85	·01262	400	7 95	·01300
	Mean	·01261		Mean	·01304

Mean value of $x/N = \frac{1}{2}(0\ 01261 + 0\ 01304) = 0{\cdot}01282$ cm gm.$^{-1}$.

Hence, by (5), $RO = Mx/N = 1455 \times 0\ 01282 = 18\ 65$ cm.

When the telescope was used, the reading of the steady point on the scale was found to be 18·7 cm, while the reading of the knife-edge on the scale was 0 0 cm. Hence $RO = 18{\cdot}7$ cm.

The three methods give for RO the values 18 13, 18·65 and 18·7 cm. The first of these is probably less accurate than the others, on account of the difficulty of obtaining sufficiently accurate values of the two periodic times.

EXPERIMENT 12. Determination of the radius of curvature of a concave mirror by the oscillations of a sphere rolling in it.

63. Introduction. If a steel sphere be displaced from its position of equilibrium on the surface of a concave spherical surface

of metal or glass, and if it be then set free, it will *roll* backwards and forwards, since the coefficient of friction between the sphere and the surface is sufficiently great to ensure that there shall be no *sliding*, at least when the oscillations are small. By observing the periodic times of the small oscillations of the sphere in the mirror, when the centre of the sphere moves in a fixed vertical plane, the radius of curvature of the mirror can be found.

A concave mirror or the concave face of a large lens forms a convenient surface for the experiment. The mirror or lens should be fixed in a block of wood, which should stand, without shake, upon a firm table. It is important that the mirror should be *firmly* attached to the table, since any slight rocking of the mirror while the sphere is oscillating leads to a rapid damping of the oscillations and makes it impossible to observe enough oscillations to obtain an accurate value of the periodic time.

To allow the radius of curvature of the surface on which the sphere rolls to be measured optically, the back of the lens should be coated with black varnish; in the case of a silvered mirror, the silvering is removed before the varnish is applied. There will then be practically no reflexion from the back of the lens or mirror, and there will be no uncertainty as to whether it is the front or the back surface which is responsible for producing the image observed in the optical method.

Highly finished hard steel spheres suitable for this experiment can be obtained from the Auto-Machinery Co. Limited, Coventry, and from other makers. They are used in ball-bearings. Better results will be obtained with large spheres than with small ones.

64. Theory of the method. Let O (Fig. 26) be the centre of curvature of the mirror, Q the centre of the sphere, A the point on the mirror vertically below O, and let P be the point of contact of the sphere with the mirror. Let the angle POA be θ radians, let the radius of the mirror be R cm. and that of the sphere be r cm.

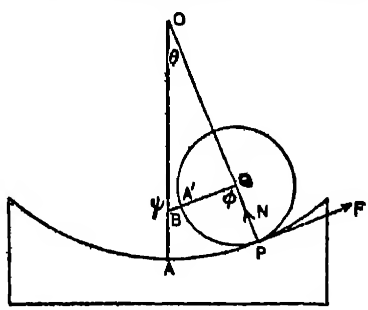

Fig. 26.

If A' be the point on the

sphere which is in contact with A when the sphere is in its lowes position, the arc AP is equal in length to the arc $A'P$, since th sphere *rolls* and does not slide. Hence, if the angle $A'QP$ be ϕ we have

$$R\theta = r\phi. \quad \text{.........................(1)}$$

If QA' cut OA in B and if the angle QBO be ψ, then ψ is th angle through which the sphere has turned since it left its lowes position, for the line QA' is fixed in the sphere.

Now $$\psi = \phi - \theta = R\theta/r - \theta$$

so that $$\theta = \frac{r}{R-r}\psi. \quad \text{..............................(2)}$$

The periodic time is easily calculated when we know (i) th potential energy of the rolling sphere in terms of ψ^2 and (ii) th kinetic energy of the sphere in terms of ω^2, where ω radians pe sec. is the angular velocity of the sphere, so that ω is the rate o increase of ψ with respect to the time, or $\omega = d\psi/dt$.

The distance of Q below O is $(R-r)\cos\theta$ and when the spher is in its lowest position the distance is $R-r$ cm. Hence, if th potential energy be reckoned zero for the lowest position of th sphere,

$$\text{Potential energy} = Mg\,(R-r)\,(1-\cos\theta)$$
$$= 2Mg\,(R-r)\sin^2\tfrac{1}{2}\theta \text{ ergs},$$

where M gm. is the mass of the sphere.

When θ is very small, $\sin\frac{1}{2}\theta$ may be replaced by $\frac{1}{2}\theta$ and then

$$\text{Potential energy} = \tfrac{1}{4}Mg\,(R-r)\,\theta^2,$$

or, by (2),

$$\text{Potential energy} = \frac{1}{4}\frac{Mgr^2}{R-r}\psi^2. \quad \text{...............(3)}$$

The kinetic energy can be written down at once, if we notice that, since the point P of the sphere is instantaneously at rest, the sphere is at that instant turning about an axis through P perpendicular to the plane AOP. Hence*, if K gm. cm.^{2} be the moment of inertia of the sphere about this axis, and if ω be the angular velocity of the sphere, i.e. the angular velocity of QA', we have

$$\text{Kinetic energy} = \tfrac{1}{2}K\omega^2 \text{ ergs}.$$

Now, the moment of inertia of the sphere† about an axis through

* *Experimental Elasticity*, Note IV. † *Ibid.*

its centre Q parallel to the axis through P is $\frac{2}{5}Mr^2$, while by the theorem of parallel axes* the moment of inertia about the axis through P exceeds $\frac{2}{5}Mr^2$ by $M.PQ^2$ or Mr^2. Thus

$$K = \tfrac{7}{5}Mr^2.$$

Hence $\qquad$ Kinetic energy $= \frac{1}{2}.\frac{7}{5}Mr^2.\omega^2$ ergs. ....(4)

We have now found expressions for the potential and kinetic energies in the forms required. We see from (3) and (4) that

$$\frac{\text{Kinetic energy for unit angular velocity}}{\text{Potential energy for unit angular displacement}} = \frac{7}{5}.\frac{R-r}{g}.$$

Hence, by § 17, if the periodic time of the small oscillations of the sphere be T secs.,

$$T = 2\pi\sqrt{\frac{7(R-r)}{5g}}. \qquad \text{..................(5)}$$

Thus the length of the simple equivalent pendulum is $\frac{7}{5}(R-r)$. Finding R from (5), we have

$$R = r + \frac{5T^2g}{28\pi^2}. \qquad \text{.......................(6)}$$

In obtaining this result the sphere has been supposed to roll and not to slide. It is shown in § 65 that this condition will be fulfilled if μ, the coefficient of friction for the sphere and the surface, be greater than $\frac{2}{7}\tan\alpha$, where α is the greatest angle between OP and the vertical.

65. Reaction between sphere and mirror. We can easily determine the least value of the coefficient of friction which will ensure that the sphere shall not slide. Let α be the greatest value of θ. Then, when $AOP = \alpha$, the sphere is instantaneously at rest and has no kinetic energy. When $AOP = \theta$, the centre of the sphere has descended the distance $(R-r)(\cos\theta - \cos\alpha)$ and hence the loss of potential energy is $Mg(R-r)(\cos\theta - \cos\alpha)$. This is equal to the kinetic energy of the sphere when $AOP = \theta$, and thus, by (4), has the value $\frac{1}{2}.\frac{7}{5}Mr^2.\omega^2$. But $\omega = d\psi/dt$ and $\psi = (R-r)\theta/r$, and hence, after slight simplification,

$$\tfrac{1}{2}.\tfrac{7}{5}(R-r)\left(\frac{d\theta}{dt}\right)^2 = g(\cos\theta - \cos\alpha). \qquad \text{............(7)}$$

* *Experimental Elasticity*, Note IV.

Differentiating with respect to t and omitting the factor $d\theta/dt$ on each side, we have

$$\tfrac{7}{5}(R-r)\frac{d^2\theta}{dt^2} = -g\sin\theta. \quad \text{.................}(8)$$

Let F be the tangential force exerted by the mirror on the sphere in the direction in which θ increases, and let N be the normal force in the direction PO. Since the acceleration of the centre of gravity of the sphere is the same as if all the forces acted there*, we have for the accelerations parallel to N and F respectively,

$$M(R-r)\left(\frac{d\theta}{dt}\right)^2 = N - Mg\cos\theta, \text{..............}(9)$$

$$M(R-r)\left(\frac{d^2\theta}{dt^2}\right) = F - Mg\sin\theta. \quad \text{............}(10)$$

From (8) and (10) we find

$$F = \tfrac{2}{7} Mg\sin\theta \text{ ....}(11)$$

And from (7) and (9) we find

$$N = \tfrac{1}{7} Mg\,(17\cos\theta - 10\cos\alpha) \text{ ....}(12)$$

The greatest value of F is $\frac{2}{7} Mg\sin\alpha$. Since $\cos\theta$ diminishes as θ increases, the least value of N occurs when $\theta = \alpha$, and then $N = Mg\cos\alpha$. Hence, the greatest value of F/N is $\frac{2}{7}\tan\alpha$. If μ, the coefficient of friction, exceeds $\frac{2}{7}\tan\alpha$ the sphere will always roll and never slide

If the motion is to be so nearly harmonic that the periodic time does not differ from that given by (5) by more than one part in 1000, we see, by Note II, that α^2 must not exceed 16/1000, and hence α must not exceed 0·1265 radian or 7° 15′. With this value of α, there will be no slipping if μ exceeds 0·0363.

66. Experimental details. In order that the vibration may continue as long as possible and also to avoid damage to the surface of the glass, the sphere and also the mirror should be carefully cleaned so as to be free from grit. The sphere should be smeared with vaseline when not in use, to keep it from rusting.

The time occupied by 100 complete vibrations of the sphere should be observed at least three times. It is convenient to fix a pointer so that it is just above the sphere when the sphere is in

* *Experimental Elasticity*, Note III.

its equilibrium position. The pointer then serves as an index and enables the observer to judge accurately the instant when the sphere passes through its equilibrium position.

In starting the sphere, care must be taken that, when the sphere is released, its centre moves in the vertical plane through the centre of curvature of the mirror and the initial position of the centre of the sphere. A slight motion at right angles to that plane will cause the centre of the sphere to describe a curve round the vertical line through the lowest point of the mirror instead of moving in a fixed vertical plane.

The diameter of the sphere may be found by calipers fitted with a vernier. If the diameter of the steel sphere as given by the makers is known, this value may be employed, as it is probably more accurate than that which could be found by the calipers.

The value of R is then calculated by equation (6) using the observed values of T and r.

The result may be tested by measuring the radius of the mirror by a spherometer, or by an optical method. In the latter method the mirror is mounted so that AO is horizontal, and a pin, which should be well illuminated, is adjusted so that the image of its tip coincides with the tip itself. The adjustment is tested by moving the eye from side to side, and is complete when the tip and its image are seen in coincidence for all positions of the eye. The tip of the pin is then at the centre of the mirror, and the distance of the tip from the mirror is equal to the radius of curvature of the mirror. The optical method will probably give the more accurate result unless care be taken in using the spherometer.

67. Practical example. The following results were obtained in an experiment in which a steel sphere $1\frac{1}{2}$ inches in diameter was used.

Radius of sphere $= r = \frac{1}{2} \times 1{\cdot}5 \times 2{\cdot}54 = 1{\cdot}905$ cm.

Observations for the periodic time gave 100 vibrations in 112·2, 112·0, 111·9 seconds. Mean value $= 112{\cdot}0$ seconds.

Hence, periodic time $= T = 1{\cdot}120$ seconds.

Acceleration due to gravity $= g = 981{\cdot}2$ cm. sec.$^{-2}$.

Hence, by (6), we have for R, the radius of the mirror,

$$R = r + \frac{5T^2 g}{28\pi^2} = 1{\cdot}905 + 22{\cdot}27 = 24{\cdot}18 \text{ cm.}$$

Two observations for R were made by the optical method. Each gave

$$R = 24{\cdot}21 \text{ cm.}$$

EXPERIMENT 13. **Determination of "g" by the oscillations of a rod rolling on a cylinder.**

68. **Introduction.** In this experiment a straight rod is placed on a horizontal circular cylinder, the axis of the rod being horizontal and in a vertical plane at right angles to the axis of the cylinder. If the rod be slightly displaced in this plane it will, when released, perform vibrations which will be harmonic if the arc of vibration be very small. From the periodic time and from the radius of the cylinder and the dimensions of the rod, the value of the gravitational acceleration can be found.

Let the circle in Fig. 27 be the section of the cylinder by the plane containing the axis of the rod. Let OA be the vertical through the centre O. Let G be the centre of gravity of the rod and GB the perpendicular from G upon the side of the rod which is in contact with the cylinder. We shall suppose that the rod is horizontal in its equilibrium position and that the rod rolls on the cylinder and does not slide; hence, if P be the point of contact at any time, the arc AP is equal to the distance BP. Then, if the radius of the cylinder be r cm. and if $AOP = \theta$, we have

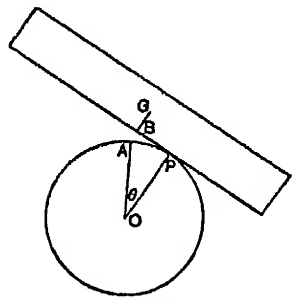

Fig. 27.

$$BP = r\theta.$$

We will now calculate the potential energy of the rod in the displaced position, the potential energy in the equilibrium position being reckoned as zero. In the displaced position the height of the centre of gravity of the rod above O is

$$(OP + BG)\cos\theta + BP\sin\theta.$$

Then, if $GB = h$ cm., the height of G above O is

$$(r + h)\cos\theta + r\theta\sin\theta.$$

In the equilibrium position the height of G above O is $r + h$, and hence we find, if the mass of the rod is m gm.,

$$\text{Potential energy} = mg\,\{(r + h)(\cos\theta - 1) + r\theta\sin\theta\}.$$

Since $\cos\theta = 1 - \frac{1}{2}\theta^2 + \frac{1}{24}\theta^4 - \ldots$, $\sin\theta = \theta - \frac{1}{6}\theta^3 + \ldots$, we have, as far as θ^2,

$$\text{Potential energy} = \tfrac{1}{2} mg(r-h)\,\theta^2 \text{ ergs.} \quad \ldots\ldots(1)$$

We see from (1) that the potential energy in the displaced position is *positive* as long as r is greater than h, and in this case the equilibrium will be *stable*. If r is less than h, the potential energy in the displaced position will be negative and consequently the equilibrium will be unstable.

Since the point of the rod in contact with the cylinder is instantaneously at rest, we have

$$\text{Kinetic energy} = \tfrac{1}{2} K\omega^2 \text{ ergs,}$$

where K is the moment of inertia of the rod about an axis through P perpendicular to the plane of vibration and ω radians per sec. is the angular velocity of the rod, i.e. the rate of increase of θ. If the moment of inertia of the rod about a parallel axis through G be mk^2, the moment of inertia about the axis through P is, by the theorem of parallel axes*,

$$K = mk^2 + m\,.\,PG^2 = m(k^2 + h^2 + r^2\theta^2).$$

Hence $\qquad$ Kinetic energy $= \frac{1}{2}m(k^2 + h^2 + r^2\theta^2)\,\omega^2$.

When the arc of vibration is infinitesimal, we can neglect $r^2\theta^2$ in comparison with $k^2 + h^2$, and then we have

$$\text{Kinetic energy} = \tfrac{1}{2} m(k^2 + h^2)\,\omega^2 \text{ ergs.} \quad \ldots\ldots\ldots(2)$$

From (1) and (2) we have

$$\frac{\text{Kinetic energy for unit angular velocity}}{\text{Potential energy for unit angular displacement}} = \frac{k^2 + h^2}{g(r-h)}.$$

Hence, by § 17, we find for the periodic time

$$T = 2\pi\sqrt{\frac{k^2 + h^2}{g(r-h)}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(3)$$

If the rod be rectangular and of length $2l$ cm. and of thickness $2a$ cm., we have

$$mk^2 = \tfrac{1}{3} m(l^2 + a^2), \quad \ldots\ldots\ldots\ldots\ldots\ldots\ldots(4)$$

and hence (3) becomes

$$T = 2\pi\sqrt{\frac{l^2 + a^2 + 3h^2}{3g(r-h)}}. \quad \ldots\ldots\ldots\ldots\ldots\ldots(5)$$

* *Experimental Elasticity*, Note IV.

From this we find for gravity

$$g = \frac{4\pi^2 (l^2 + a^2 + 3h^2)}{3(r-h)T^2} \quad\text{.....................(6)}$$

69. Experimental details. A straight bar of tool steel of rectangular section is suitable for the experiment; it must be so stiff that it does not bend appreciably under its own weight when it is supported on the cylinder. The part of its surface which rests upon the cylinder should be plane and also free from rust. The cylinder may be any metal cylinder with a smooth surface; a hardened steel cylinder would be best. It should be firmly fixed with its axis horizontal. The rod is laid on the cylinder and is adjusted so that in the equilibrium position it is horizontal and also perpendicular to the axis of the cylinder. An indicator is set up near the rod and the passage of the rod past the indicator is used as the timing signal. The periodic time, T secs, is deduced from observations of the time occupied by a number of complete vibrations, this number should be as large as the decay of the vibrations permits. At least two or three independent observations of T should be made. Since the periodic time increases with the amplitude, the amplitude should be small (see Note III).

Unless the surfaces are hard and true no real advantage will be gained by making the amplitude very small, for this is apt to make the periodic time depend upon the imperfections of the surfaces in the immediate neighbourhood of the line of contact.

Unless the rod be very nearly straight, it will be difficult to obtain an accurate value for h. If the length, $2l$, be large compared with the thickness, we may neglect h^2 in comparison with k^2 in (3), but we may not be able to neglect h in comparison with r. The difficulty may be met by taking two sets of observations with first one face and then the opposite face of the bar in contact with the cylinder. If h_1, h_2 be the two perpendicular distances from G upon these two faces and if T_1, T_2 be the corresponding periodic times, we have, by (3), when h_1^2 and h_2^2 can be neglected in comparison with k^2

$$r - h_1 = \frac{4\pi^2 k^2}{gT_1^2}, \quad r - h_2 = \frac{4\pi^2 k^2}{gT_2^2}.$$

It will be sufficiently accurate to take $k^2 = \frac{1}{3}l^2$ in (4) and then, since $h_1 + h_2 = 2a$, the thickness of the rod, we have

$$2r - 2a = \frac{4\pi^2 l^2}{3g}\left(\frac{1}{T_1^2} + \frac{1}{T_2^2}\right),$$

and thus
$$g = \frac{4\pi^2 l^2}{3(2r - 2a)}\left(\frac{1}{T_1^2} + \frac{1}{T_2^2}\right). \qquad \text{..............(7)}$$

It has been assumed that the parts of the faces of the rod which are in contact with the cylinder are plane or that at least their radii of curvature are very large compared with r.

The method may be tested by vibrating the bar on different cylinders and comparing the results.

70. Practical example. The results may be entered as in the following record of an experiment by Miss B. G. E. Hooke.

A bar of tool steel was used

Length of bar $= 2l = 100{\cdot}64$ cm. Hence $l = 50{\cdot}32$ cm.

Thickness of bar $= 2a = 0{\cdot}89$ cm.

The width of the bar was 1·85 cm. It was found by balancing on a knife-edge that the centre of gravity was within one mm of the centre of the bar. The cross-section of the bar was therefore probably very nearly uniform.

The bar was made to vibrate on the edge of a cast-iron flywheel for which $2r = 22{\cdot}80$ cm.

When the side (1) of the bar was in contact with the cylinder, the times of 100 vibrations were 177·0, 176·4, 176·8 secs.

Hence periodic time $= T_1 = 1{\cdot}767$ secs.

When the side (2) was in contact with the cylinder, the times of 100 vibrations were 176·4, 177·0, 176·0 secs.

Hence periodic time $= T_2 = 1{\cdot}765$ secs.

The periodic times were so nearly equal that each was put equal to the mean value 1·766 secs. in formula (7).

Then $$g = \frac{8\pi^2 l^2}{3(2r - 2a)T^2} = \frac{8\pi^2 \times 50{\cdot}32^2}{3 \times 21{\cdot}91 \times 1{\cdot}766^2} = 975 \text{ cm. sec.}^{-2}.$$

A similar set of observations was made by G. F. C. Searle for the same bar vibrating on a brass cylinder for which $2r = 11{\cdot}44$ cm. It was found that $T_1 = 2{\cdot}469$ secs., $T_2 = 2{\cdot}466$ secs. The experiment gave $g = 1037$ cm. sec.$^{-2}$.

EXPERIMENT 14. **Study of a vibrating system with two degrees of freedom.**

71. Theory of the experiment. In many interesting examples the system requires more than a single coördinate to fix its configuration, and to illustrate the principles involved in

these cases we shall consider the problem of the motion of a system having two degrees of freedom. Out of the many examples of such systems we select one which has the advantage that it is capable of accurate and detailed experimental examination.

Consider a system consisting of two inertia bars H, K (Fig. 28) clamped to a vertical wire stretched between two fixed clamping screws A and C, and suppose (though the restriction is unnecessary and is only made for simplicity of statement) that when the bars are at rest the wire is everywhere free from torsion. If the motion of each bar be simply a rotation about the axis of the vertical wire, two coördinates are sufficient to fix the configuration of the system, these being the angles θ and ϕ which the bars H, K make with their equilibrium positions. It is convenient to take the same direction as positive for both θ and ϕ.

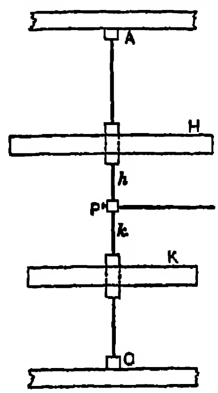

Fig. 28.

If, now, the bars be displaced from their equilibrium positions through the angles θ and ϕ, the top part of the wire exerts a couple on H proportional to θ and the middle part of the wire a couple proportional to $\theta-\phi$, each tending to turn H in the negative direction. In the same way, the bar K experiences a couple proportional to ϕ and a couple proportional to $\phi-\theta$, each tending to turn K in the negative direction. The middle part of the wire exerts equal and opposite couples upon the two bars. When the lengths and radii of the three portions of the wire and the rigidity of the material are known, the couples can be calculated, but for the present experiment it is sufficient to take the two couples acting on H in the positive direction as

$$-a\theta \text{ and } -b(\theta-\phi),$$

and the two couples acting on K in the positive direction as

$$-c\phi \text{ and } -b(\phi-\theta).$$

It will be noticed that the quantities a, b and c are essentially *positive*.

If F and G be the couples acting on H and K,

$$F=-(a+b)\theta+b\phi, \qquad G=-(c+b)\phi+b\theta,$$

and hence, if the moments of inertia of the two bars about the vertical axis of the wire be H and K, we have*

$$\frac{d^2\theta}{dt^2}=\frac{F}{H}=-\frac{a+b}{H}\theta+\frac{b}{H}\phi, \qquad \frac{d^2\phi}{dt^2}=\frac{G}{K}=-\frac{c+b}{K}\phi+\frac{b}{K}\theta,$$

or

$$\left(\frac{d^2}{dt^2}+\frac{a+b}{H}\right)\theta=\frac{b}{H}\phi, \quad \text{.................} \quad (1)$$

$$\left(\frac{d^2}{dt^2}+\frac{c+b}{K}\right)\phi=\frac{b}{K}\theta. \quad \text{.................} \quad (2)$$

We see that the moments of inertia of the two bars and the elastic constants of the three portions of wire are involved through the quantities

$$\frac{a+b}{H}, \quad \frac{c+b}{K}, \quad \frac{b}{H}, \quad \frac{b}{K}.$$

It will therefore be convenient, before going further, to express these quantities in terms of the frequencies of four vibrations which can be directly observed. The frequency, which is measured by the number of complete vibrations per second, is selected in preference to the periodic time in order to save space.

If K be held fixed in its equilibrium position, so that $\phi=0$, the couple acting on H in the positive direction is $-(a+b)\theta$, and thus the equation of motion for H is

$$\frac{d^2\theta}{dt^2}=-\frac{a+b}{H}\theta.$$

Hence, if t_1 be the periodic time, and n_1 be the frequency, so that $n_1=1/t_1$, we have, by § 10,

$$t_1^2=\frac{1}{n_1^2}=4\pi^2\frac{H}{a+b},$$

or

$$4\pi^2 n_1^2=\frac{a+b}{H}. \quad \text{.........................} \quad (3)$$

Similarly, if n_2 be the frequency of K when H is held fixed in its equilibrium position,

$$4\pi^2 n_2^2=\frac{c+b}{K}. \quad \text{.........................} \quad (4)$$

Again, let the top and bottom parts of the wire be removed, the middle part between H and K remaining, and let n_3 be the

* *Experimental Elasticity*, Note III.

frequency of H when K is fixed, and n_4 the frequency of K when H is fixed (In each of these two cases the vibrating bar must be *below* the fixed one so as to be *suspended* by the wire.)

Then $$4\pi^2 n_3^2 = \frac{b}{H}, \quad 4\pi^2 n_4^2 = \frac{b}{K}. \quad \text{..............(5)}$$

When these values are used in (1) and (2), the equations of motion become

$$\left(\frac{d^2}{dt^2} + 4\pi^2 n_1^2\right)\theta = 4\pi^2 n_3^2 \phi, \quad \text{...............(6)}$$

$$\left(\frac{d^2}{dt^2} + 4\pi^2 n_2^2\right)\phi = 4\pi^2 n_4^2 \theta. \quad \text{...............(7)}$$

Hence

$$\left(\frac{d^2}{dt^2} + 4\pi^2 n_2^2\right)\left(\frac{d^2}{dt^2} + 4\pi^2 n_1^2\right)\theta = \left(\frac{d^2}{dt^2} + 4\pi^2 n_2^2\right) 4\pi^2 n_3^2 \phi$$
$$= 16\pi^4 n_3^2 n_4^2 \theta,$$

or $$\frac{d^4\theta}{dt^4} + 4\pi^2 (n_1^2 + n_2^2)\frac{d^2\theta}{dt^2} + 16\pi^4 (n_1^2 n_2^2 - n_3^2 n_4^2)\theta = 0. \ldots(8)$$

It is clear that ϕ satisfies exactly the same differential equation as θ.

To solve (8) we assume that H performs harmonic vibrations with frequency N and we therefore write

$$\theta = A \sin(2\pi N t + \epsilon), \quad \text{.....................(9)}$$

where ϵ is a constant.

If, when a proper value is chosen for N, this value of θ satisfies (8) our assumption will be justified. On differentiating (9) we find

$$\frac{d^2\theta}{dt^2} = -4\pi^2 N^2 \theta, \quad \frac{d^4\theta}{dt^4} = 16\pi^4 N^4 \theta.$$

Substituting these values in (8) and dividing by $16\pi^4\theta$, we obtain a quadratic equation for N^2. Thus

$$N^4 - (n_1^2 + n_2^2) N^2 + n_1^2 n_2^2 - n_3^2 n_4^2 = 0. \quad \text{.........(10)}$$

Solving this equation we find

$$N^2 = \tfrac{1}{2}(n_1^2 + n_2^2) \pm \left[\tfrac{1}{4}(n_1^2 + n_2^2)^2 - (n_1^2 n_2^2 - n_3^2 n_4^2)\right]^{\frac{1}{2}}$$
$$= \tfrac{1}{2}(n_1^2 + n_2^2) \pm \left[\tfrac{1}{4}(n_1^2 - n_2^2)^2 + n_3^2 n_4^2\right]^{\frac{1}{2}}. \quad \text{..............(11)}$$

The second expression shows that both values of N^2 are *real* since

the quantity under the square root is *positive*. By (3), (4) and (5) we find that

$$4(n_1^2 n_2^2 - n_3^2 n_4^2) = (ac + ab + cb)/4\pi^4 HK.$$

Thus the quantity on the left is positive and hence we see by the first expression for N^2 that both values of N^2 are positive Thus there are two values of N^2 which are positive and real, each corresponding to an oscillatory motion.

Let N_1^2, N_2^2 be the two roots of (10) and let N_1^2 be the greater root, corresponding to the positive sign of the square root in (11). Then the motion of H is represented by*

$$\theta = A_1 \sin(2\pi N_1 t + \epsilon_1) + A_2 \sin(2\pi N_2 t + \epsilon_2). \quad \ldots\ldots(12)$$

Substituting this value of θ in (6) we find that the motion of K is given by

$$\phi = \frac{n_1^2 - N_1^2}{n_3^2} A_1 \sin(2\pi N_1 t + \epsilon_1) + \frac{n_1^2 - N_2^2}{n_3^2} A_2 \sin(2\pi N_2 t + \epsilon_2). \quad \ldots(13)$$

From (12) and (13) we see that, since A_1, A_2, ϵ_1 and ϵ_2 are arbitrary, each bar has two simultaneous simple harmonic motions of arbitrary amplitude and phase, superposed the one on the other. We also see, what is most important, that the phase of each vibration for K is the same as that of the corresponding vibration for H or differs from it by exactly half a period, and that for either pair of corresponding vibrations the amplitudes of H and K have a constant ratio.

If we substitute the value of ϕ given by (13) in (7) and note that both N_1^2 and N_2^2 satisfy (10), we recover the value of θ given by (12).

Let us now examine the sign of the quantity

$$(n_1^2 - N_1^2)(n_1^2 - N_2^2).$$

* Though there are *four* values of N, viz. $\pm N_1$ and $\pm N_2$, there are only *two* frequencies. For, if θ_1 be the part of θ depending on N_1, the most general expression for θ_1 is

$$\theta_1 = B_1 \sin(2\pi N_1 t + \eta_1) + B_2 \sin(-2\pi N_1 t + \eta_2)$$

and this may be reduced to $A_1 \sin(2\pi N_1 t + \epsilon_1)$, where A_1 and ϵ_1 are arbitrary since B_1, B_2, η_1 and η_2 are arbitrary.

By the theory of equations, we have, from (10)

$$N_1^2 N_2^2 = n_1^2 n_2^2 - n_3^2 n_4^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(14)$$

$$N_1^2 + N_2^2 = n_1^2 + n_2^2, \quad \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

and thus we find

$$(n_1^2 - N_1^2)(n_1^2 - N_2^2) = -n_3^2 n_4^2.$$

Hence, one of the two factors on the left is positive while the other is negative, and, since $N_1^2 > N_2^2$, the factor $n_1^2 - N_1^2$ is the negative one.

Thus, by (12) and (13), we see that for the vibration of greater frequency (N_1) the displacements of the two bars are always in opposite directions, and that for the vibration of smaller frequency (N_2) the displacements are always in the same direction.

Though it is theoretically possible to set the system in motion so that either A_1 or A_2 is zero, it would be difficult to do this practically But it is easy to give the system such a motion that one coefficient—say A_1—is large while the other is small. The intervals between the transits of H (or K) through its equilibrium position will then be nearly constant, and the average of a large number of intervals will be the same as if the main vibration had alone existed.

If a light pointer P (Fig. 28) be attached to the middle part of the wire by a small clamp as in EXPERIMENT 7, it will be possible to adjust the position of the clamp so that the pointer does not move when the vibration corresponding to N_1 alone exists, since the bars are then vibrating in opposite directions.

The most general motion of the system is obtained by superposing a vibration of frequency N_2 upon that of frequency N_1. In this general motion, which may be obtained by starting the system in any way, the bars will move in an apparently very irregular manner, but the pointer will perform simple harmonic vibrations with frequency N_2 when it is clamped in the proper position. In § 72 it is shown how this pointer and clamp may be used to produce either a motion in which A_1 is nearly zero or a motion in which A_2 is nearly zero.

The theory may be tested by observing the frequencies n_1, n_2, n_3 and n_4 and comparing the values of N_1 and N_2 obtained from (11) with the values found by direct observation. The formulae (14) and (15) can also be tested.

If the part of the wire below K be removed altogether, then $c=0$ and, consequently, by (4) and (5), $n_4=n_2$. This is the case in the experiment described in § 72.

If a long piece of plaited silk be substituted for the wire above H, its torsional couple is negligible and so we may put $a=0$, and then, by (3) and (5), $n_3=n_1$. If, further, the wire below K be removed, $c=0$ and $n_4=n_2$. This case corresponds to EXPERIMENT 7.

72. Experimental details. In the following experiment two inertia bars H, K are suspended by a wire from a support A as in Fig. 29. The wire is not continued below the bar K.

The system can be taken to pieces and be put together again without causing any change in the constants, if the two parts of the wire are soldered into brass cylinders 3 or 4 cm. in length and 5 mm. in diameter, as in Fig. 29. These cylinders fit into holes bored in the bars and are held fast by small screws. The torsional couples are then (practically) independent of the positions of the bars on these cylinders and of the position of the top cylinder in the clamp which holds it. The reader may refer to the remarks on inertia bars in *Experimental Elasticity*, Note VII.

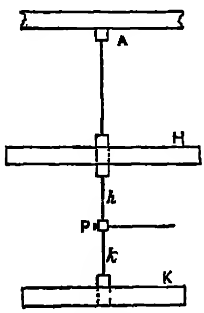

Fig 29.

To give the system a motion in which the vibration of greater frequency (N_1) predominates, the bars are turned in opposite directions and are then set free. After a few trials, in which the ratio of the angles is varied, the bars can be made to vibrate in such a manner that they move in opposite directions and both come to rest at practically the same times. To obtain the vibration of smaller frequency (N_2), the bars are turned in the same direction, the lower bar being turned through the greater angle. After the system has been set into vibration, any pendulum motion of the bars should be carefully destroyed.

When the vibration of greater frequency (N_1) alone exists, a critical position can be found for the pointer P, such that it does not vibrate. To test the adjustment, the bars are given a motion in which the vibration of greater frequency predominates. If, now,

after the pointer has been brought to rest, it remains at rest when set free, the adjustment is complete, and the motion of the bars, after the pointer has been brought to rest, is simply a vibration of frequency N_1. If the clamp be held fast in the fingers while the bars vibrate, it will exert a couple upon them, unless it is in the critical position. From the direction of this couple it is easy to tell in which direction the clamp should be moved to obtain more perfect adjustment.

When the pointer has been approximately adjusted to the critical position, we can easily obtain approximately pure vibrations corresponding to either N_1 or N_2. To obtain the vibration of greater frequency the bars are turned in opposite directions and are then set free; if the pointer be then brought to rest, the motion will be approximately a vibration of frequency N_1. To obtain the vibration of smaller frequency, the bars are brought to rest and a couple is then applied by the fingers to the clamp carrying the pointer so as to set the system in motion; the motion is then approximately a vibration of frequency N_2.

When the motion is a pure vibration of frequency N_1, we have, by (12) and (13),

$$\phi/\theta = (n_1^2 - N_1^2)/n_2^2,$$

so that, if the critical position of the pointer P divides the wire between H and K into two parts h, k, of which h is the upper part, then, when the wire is uniform,

$$\frac{k}{h} = -\frac{\phi}{\theta} = -\frac{n_1^2 - N_1^2}{n_2^2}.$$

When the frequencies have been found, the value of h/k calculated from this expression can be compared with that obtained from the position of the pointer on the wire. Since $n_1^2 - N_1^2$ has been shown to be negative, the ratio k/h is positive. The lengths h and k are, of course, measured from the points where the wires are soldered into the cylinders.

Each frequency is deduced from the time occupied by a considerable number of the corresponding vibrations, the time being observed *at least* twice. Thus, if 100 vibrations occupy 428·7 seconds, the frequency is 1/4·287 or 0·2333 sec.$^{-1}$. In each case the number of vibrations should be large enough to occupy not

less than three minutes. A rod should be set up close to one end of the vibrating bar when it is in its equilibrium position, and the transits of the bar past the rod should be observed

When the frequency n_1 is to be found, the bar K is held fast in a strong, massive clip, or is clamped to a heavily weighted block of wood, care being taken that the wire remains vertical.

For n_2 and n_3 the wire should be removed from the support A. In the determination of n_2 the bar H is clamped to a shelf so that K is suspended by the wire, and in the determination of n_3 the bars are interchanged so that now the bar K is clamped to the shelf while H is suspended by the wire. The loose part of the wire (the part between A and H in Fig. 29) should be *above* the bar which is clamped, and should be held in a clip to prevent it from being seriously bent.

73. Practical example. In an experiment made by Mr Wrigley the following mean values were obtained for the frequencies .—

$$\frac{1}{n_1}=3\cdot828 \text{ secs.},\quad \frac{1}{n_2}=\frac{1}{n_4}=4\cdot327 \text{ secs.},\quad \frac{1}{n_3}=6\cdot126 \text{ secs.}$$

In deducing the values of $1/N_1$ and $1/N_2$ from these three quantities a considerable number of arithmetical steps is necessary To avoid the possible accumulation of sensible errors, seven-figure logarithms were used. In this way the following values were found

$$n_1^2=0\cdot0682427,\quad n_2^2=n_4^2=0\cdot0534104,\quad n_3^2=0\cdot0266469.$$

From the equation (11)

$$N^2=\tfrac{1}{2}(n_1^2+n_2^2)\pm[\tfrac{1}{4}(n_1^2-n_2^2)^2+n_3^2n_4^2]^{\frac{1}{2}}$$

we obtain
$$N^2=0\cdot0608266\pm0\cdot0384476,$$

and thus
$$N_1^2=0\cdot0992742,\quad N_2^2=0\cdot0223790.$$

Hence
$$\frac{1}{N_1}=3\cdot1738 \text{ secs.},\quad \frac{1}{N_2}=6\cdot6847 \text{ secs.}$$

By actual observation of the two vibrations it was found that

$$\frac{1}{N_1}=3\cdot174 \text{ secs.},\quad \frac{1}{N_2}=6\cdot684 \text{ secs.}$$

There is thus very close agreement between the calculated and observed values of the frequencies.

EXPERIMENT 15. **Determination of "g" by the oscillations of a cone rolling on an inclined plane.**

74. **Introduction.** The cone is supposed to be a cone of revolution about an axis of symmetry. The cone need not be of uniform density provided it be in all respects symmetrical about the axis, so that its centre of gravity lies on the axis. If the cone be placed on an inclined plane or "track," it will, if the slope of the track be not too great, be in equilibrium in either of the two positions in which the vertical plane through the axis intersects at right angles the straight line in which the track cuts a horizontal plane. The equilibrium will, however, be *stable* only in that position in which the projection of the centre of gravity of the cone on the track is lower than the vertex.

It is assumed that the coefficient of friction is sufficient to ensure both that the cone rolls and does not slip, and that the cone does not slide down the plane. It is further assumed that the slope of the plane is sufficiently small to ensure the possibility of stable equilibrium.

To find the periodic time, we apply the methods of §§ 16, 17 and calculate the kinetic energy of the cone when rolling and its potential energy when it is displaced from its position of stable equilibrium.

75. **Kinetic energy of cone.** Let O (Fig. 30) be the vertex and OG the axis of the cone. If the cone roll, the vertex O remains at rest, and thus O is a fixed point. Through O draw OR perpendicular to the track. Then the centre of gravity G describes an arc of a circle about OR as axis. The radius of this circle is equal to ON, where GN is the perpendicular from G upon the track. Let $OG = h$, $GON = \alpha$, and let the angular velocity of G about OR at any time be ω. Then the linear velocity of G is $\omega \, . \, ON$.

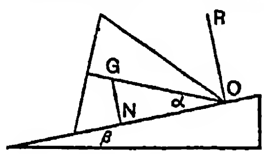

Fig. 30.

Since the generating line ON is instantaneously at rest, the linear velocity of G is also equal to $\Omega \, . \, GN$, where Ω is the angular velocity of the cone about the line of contact. Hence

$$\Omega \, . \, GN = \omega \; ON,$$

or

$$\Omega = \omega \, . \, ON/GN = \omega \, . \cot \alpha \qquad (1)$$

If the moment of inertia of the cone about a generating line such as ON be K, and if T be the kinetic energy of the cone, we have $T = \frac{1}{2}K\Omega^2$, and thus

$$T = \tfrac{1}{2}K\Omega^2 = \tfrac{1}{2}K\cot^2\alpha \,.\, \omega^2. \qquad (2)$$

76 Potential energy of cone. Let OS (Fig. 31) be the line of greatest slope on the track and let OS make an angle β with a horizontal plane FXZ. Then the line of contact ON_0 in the position of stable equilibrium lies along OS. In any other position of the cone, let the line of contact ON make an angle θ with ON_0, and let NH be the perpendicular from N on ON_0, thus $N_0H = ON(1-\cos\theta)$. Then the height of N above the plane FXZ exceeds that of N_0 by $ON(1-\cos\theta)\sin\beta$. But GN is perpendicular to the track FYZ and is of constant length, and hence G and N have both risen above their lowest positions by the same distance, viz. by $ON(1-\cos\theta)\sin\beta$.

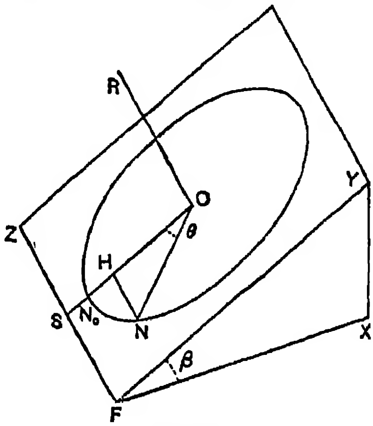

Fig. 31.

Let U be the potential energy of the cone in its displaced position, and let $U = 0$ when $\theta = 0$. Let M be the mass of the cone. Then, since $ON = h\cos\alpha$, we have

$$U = Mg \,.\, ON(1-\cos\theta)\sin\beta = Mgh\cos\alpha\sin\beta(1-\cos\theta). \quad \ldots(3)$$

When θ is small, we may write $1-\cos\theta = \frac{1}{2}\theta^2$, and thus

$$U = \tfrac{1}{2}Mgh\cos\alpha\sin\beta \,.\, \theta^2. \qquad (4)$$

77. Periodic time. Since $\omega = d\theta/dt$, the expressions for the kinetic and potential energies are in the forms discussed in § 16. Hence, if T_0 be the periodic time of small vibrations, we have, by § 17,

$$T_0 = 2\pi\sqrt{\frac{\text{kinetic energy for unit } \omega}{\text{potential energy for unit } \theta}}.$$

Hence, by (2) and (4),

$$T_0 = 2\pi\sqrt{\frac{K\cot^2\alpha}{Mgh\cos\alpha\sin\beta}} = 2\pi\sqrt{\frac{K\cos\alpha}{Mgh\sin^2\alpha\sin\beta}}. \qquad (5)$$

78. The roller. Instead of a cone, we use a roller formed of two solid steel spheres soldered together with a minimum of solder. This body can be enveloped by a circular cone and will roll just as an actual cone rolls. Let the centres of the spheres be A, B (Fig 32) and their radii a, b, and let $a > b$. Then $AD = a$, $BE = b$. Let AB, DE meet in O; then O is the vertex of the enveloping cone, and $AOD = \alpha$. Let the masses of the spheres be m_1, m_2. Then $M = m_1 + m_2$.

By the theorem of parallel axes, the moment of inertia of the body about OD is

$$K = m_1 (\tfrac{2}{5}a^2 + a^2) + m_2 (\tfrac{2}{5}b^2 + b^2) = \tfrac{7}{5}(m_1 a^2 + m_2 b^2). \quad \dots(6)$$

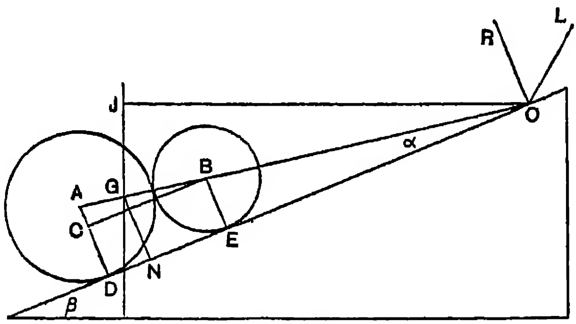

Fig. 32.

Draw BC parallel to OD to meet AD in C; then $AC = a - b$. Now $AO/AB = AD/AC$, $BO/AB = BE/AC$, and hence

$$OA = a(a+b)/(a-b), \quad OB = b(a+b)/(a-b).$$

Since G is the centre of gravity, $Mh = M \, . \, OG = m_1 \, . \, OA + m_2 \, . \, OB$.

Hence $$Mh = (m_1 a + m_2 b)(a+b)/(a-b), \quad \dots\dots\dots\dots(7)$$

and thus $$\frac{K}{Mh} = \frac{7(m_1 a^2 + m_2 b^2)(a-b)}{5(m_1 a + m_2 b)(a+b)}. \quad \dots\dots\dots\dots\dots(8)$$

From the triangle ABC, we have

$$\sin\alpha = \frac{AC}{AB} = \frac{a-b}{a+b}, \quad \cos^2\alpha = 1 - \left(\frac{a-b}{a+b}\right)^2 = \frac{4ab}{(a+b)^2}.$$

If we write $$c = K\cos\alpha/(Mh\sin^2\alpha),$$

the length c is a constant for the system of two spheres. We find

$$c = \frac{14(m_1 a^2 + m_2 b^2)(ab)^{\frac{1}{2}}}{5(m_1 a + m_2 b)(a-b)}. \quad \dots\dots\dots\dots\dots(9)$$

If the length of the simple equivalent pendulum be λ, we have, by (5),

$$\lambda = c/\sin\beta. \qquad \text{.......................(10)}$$

When the periodic time T_0 has been found, we have

$$g = 4\pi^2\lambda/T_0^2 = 4\pi^2 c/(T_0^2 \sin\beta). \qquad \text{...........(11)}$$

If the two spheres have the same density, $m_1/a^3 = m_2/b^3$, and, if $b/a = n$, then

$$c = \frac{14(a^5+b^5)(ab)^{\frac{1}{2}}}{5(a^4+b^4)(a-b)} = \frac{14a(1+n^5)n^{\frac{1}{2}}}{5(1+n^4)(1-n)}. \qquad \text{......(12)}$$

79. Conditions for stability and oscillation. If $a = b$, the roller will not *oscillate* on an inclined track. If $n = b/a$ be small, there will be no position of stable equilibrium, and hence no possibility of oscillation, unless the slope be very small.

By Fig. 32, we see that stable equilibrium will be possible until, as the slope of the track is increased, GD becomes vertical. If ϵ be the critical value of β, then $\epsilon = DGN$ and

$$\tan\epsilon = DN/GN = DN/(h\sin\alpha).$$

Since $DN = AG\cos\alpha$, we have $\tan\epsilon = AG\cot\alpha/h$. But

$$M \,.\, AG = m_2 \,.\, AB,$$

and hence we find, by (7), that

$$\tan\epsilon = 2m_2(ab)^{\frac{1}{2}}/(m_1 a + m_2 b).$$

It will suffice to consider the case of equal densities, in which

$$m_1/a^3 = m_2/b^3$$

Then $$\tan\epsilon = 2n^{\frac{7}{2}}/(1+n^4). \qquad \text{.......................(13)}$$

If the least length of the equivalent pendulum for given values of a and n be l_0, we have $l_0/a = \operatorname{cosec}\epsilon \,.\, c/a$. The values of l_0/a in the table (page 89) were calculated by (12) and (13).

The condition that $\beta = \epsilon$ ensures that stable equilibrium is just possible, but, unless β be less than ϵ, no oscillations (with *both* spheres touching the track throughout) of finite amplitude will be possible.

In Fig. 32, let OL be normal to the plane ROG, L lying behind that plane. Let the body roll so that A recedes from the reader. With a right-handed relation between the direction of a line and rotation about it, the line of contact OD represents the direction of the angular momentum $K\Omega$. Since GOL is a plane of symmetry,

there is no angular momentum about OL. During time dt, the line of contact moves round to OD' through ωdt towards OL and the normal to the moving plane ROG moves to OL'. At the end of dt the angular momentum about OD' is $K\{\Omega + (d\Omega/dt)\,dt\}$, and that about OL' is zero. Hence the angular momentum about the *fixed* line OL is $K\{\Omega + (d\Omega/dt)\,dt\}\sin(\omega dt)$, or, to the first power of dt, $K\Omega\omega dt$. Hence the rate of increase of angular momentum about OL is $K\Omega\omega$.

Let P_1, P_2 be the components normal to the track of the forces exerted by the track upon the spheres at D and E, when $\theta = 0$. Let OJ (Fig. 32) be the perpendicular from O upon the vertical through G. Then the moment about OL of the forces P_1, P_2, Mg is $P_1 . OD + P_2 . OE - Mg . OJ$, and this equals the rate of increase of angular momentum about OL. Hence

$$P_1 . OD + P_2 . OE = Mg . OJ + K\Omega\omega.$$

The centre of gravity has no acceleration perpendicular to the track, and hence

$$P_1 + P_2 = Mg\cos\beta.$$

Thus $$P_1 . DE = Mg(OJ - OE\cos\beta) + K\Omega\omega,$$

and $$P_2 . DE = Mg(OD\cos\beta - OJ) - K\Omega\omega.$$

If there be just no pressure at E, $P_2 = 0$, and

$$K\Omega\omega = Mg(OD\cos\beta - OJ).$$

Let the arc of vibration from side to side be 2ψ, when P_2 is just zero for $\theta = 0$. Then, when $\theta = 0$,

$$\tfrac{1}{2}K\Omega^2 = Mgh\cos\alpha\sin\beta(1 - \cos\psi).$$

But $\Omega\omega = \Omega^2\tan\alpha$ and $OJ = h\cos(\beta - \alpha)$, and thus

$$2h\sin\alpha\sin\beta(1 - \cos\psi) = OD\cos\beta - h\cos(\beta - \alpha).$$

Hence

$$\tan\beta = \frac{OD - h\cos\alpha}{h\sin\alpha(3 - 2\cos\psi)} = \frac{DN}{h\sin\alpha}\cdot\frac{1}{3 - 2\cos\psi}.$$

Thus $$\tan\beta = \tan\epsilon/(3 - 2\cos\psi). \quad \ldots \ldots\ldots\ldots(14)$$

If the value of β corresponding to $\psi = 60°$ be η, we have

$$\tan\eta = \tfrac{1}{2}\tan\epsilon. \ldots\ldots\ldots\ldots\ldots\ldots(15)$$

If l be the length of the equivalent pendulum when the roller makes *small* oscillations on the track with slope η, we have $l/a = \operatorname{cosec}\eta \,.\, c/a$.

The table shows the values of c/a, ϵ, l_0/a, η, l/a for various values of n.

n	c/a	ϵ	l_0/a	η	l/a
1·0	∞	45° 0′ 0″	∞		
0·9	25·51	39 52 8	39·80	22° 39′ 56″	66·20
0·8	11·79	33 0 50	21·65	17 59 51	38·16
0·7	7·356	24 50 11	17·51	13 1 48	32·63
0·6	5·173	16 30 3	18·22	8 25 30	35·31
0·5	3·843	9 26 46	23·42	4 45 19	46·36
0·4	2·907	4 30 48	36·94	2 15 37	73·71
0·3	2·179	1 40 50	74·29	0 50 26	148·5
0·2	1·563	0 24 33	218·8	0 12 16	438·0
0·1	0·984	0 2 10	1556	0 1 5	3123
0·0	0·000	0 0 0	∞	0 0 0	

If the force exerted by the track upon the roller in the direction DO be Q, when the roller is passing through its central position, we have

$$Q = Mg \sin\beta + M \,.\, ON\omega^2 = Mg \sin\beta + Mh\Omega^2 \cos\alpha \tan^2\alpha,$$

and thus $Q > Mg \sin\beta$, unless $\Omega^2 = 0$. If the friction cannot supply the force Q, the roller will slide. It may thus happen that, although stable equilibrium is well secured, the path of G may cease to be an arc of a *fixed* circle, on account of sliding, if the motion be not small.

The choice of n is limited by practical considerations. The coefficient of friction between the roller and the track is so small that, when $\sin\beta > 0·1$, the roller slides down the track, and thus, for values of n exceeding about 0·45, the least value of λ/a is $10c/a$. The ratio λ/a should not be large, since the effects of draughts, slight irregularities in the track, or other disturbing causes increase as λ/a increases. When n is less than about 0·45, stable equilibrium and oscillation will not be possible unless $\sin\beta$ be less than about 0·1, as column 3 shows, and column 4 will then give the least values of l/a. For accurate measurement, β should be as large as possible, and thus n should not be less than 0·5.

The motion of a roller for which n is about 0·4 is interesting. A value of β is easily reached in which, though stable equilibrium is secured, the smaller sphere breaks contact with the track when oscillations are not very small.

80. Experimental details. In order that the relation $m_1/a^3 = m_2/b^3$ may be tested, the mass of each sphere should be found and recorded before they are soldered together. A spot on each sphere is "tinned" and then the larger sphere is loosely held so that it rests on the other, the "tinned" spots being in contact. The spheres are now heated by a flame until the solder "runs." Only a small amount of solder is needed. The spheres should be kept free from rust.

The body rolls on a piece of plate glass, the "track," fixed to a heavy table fitted with levelling screws, whose tips are U, V, W. This table stands on a plane bed plate; an engineer's surface plate is convenient. After the bed plate has been levelled, the track is levelled. A block with parallel faces is then placed under the foot of the screw W, so as to tilt the table. The periodic time T_0 of *small* oscillations of the roller is then observed. The oscillation may be maintained by properly timed blasts of air supplied by bellows. A series of observations may be made in which the slope β is varied, T_0 being found for each slope. The product $T_0^2 \sin \beta$ is theoretically constant.

If the perpendicular from W on UV be z, and if the height of the block be d, then $\sin \beta = d/z$. If $VW = u$, $WU = v$, $UV = w$, and if $u + v + w = 2s$, then

$$z = (2/w)\{s(s-u)(s-v)(s-w)\}^{\frac{1}{2}}. \quad \text{.........(16)}$$

The slope may also be found by the goniometer described in §35. An auto-collimating method is employed. Since, however, the goniometer arm turns about a *vertical* axis and the track turns about a *horizontal* axis, an optical device is necessary. A plane mirror, furnished with levelling screws, and called the collimating mirror, is placed on the track and is levelled when the latter is level. The track itself may be used as the collimating mirror, if its lower surface be silvered. A second plane mirror, called the reflector, is then adjusted so that its plane (i) is perpendicular to the vertical plane through UV and (ii) makes 45° with a horizontal plane. The goniometer is placed with its axis parallel to UV, at a height corresponding to the reflector. An image of the wire of the goniometer will then be seen near the wire itself, and a slight adjustment of the arm of the goniometer will give exact coincidence. This image is formed by rays which have suffered two reflexions at the

reflector and one at the collimating mirror and have passed twice through the lens. The goniometer is adjusted so that in this zero position the arm is in its central position

The track is now tilted through angle β and carries with it the collimating mirror. The normal to the mirror turns through angle β in a vertical plane perpendicular to UV. Now, if any two rays in this *vertical* plane be incident on the reflector, the reflected rays lie in a *horizontal* plane, and the angle between the reflected rays equals that between the incident rays. Hence, if the goniometer arm be turned so that there is again coincidence between the wire and its image, the arm will have moved through β. The angle β is found from the goniometer readings as described in § 35.

A totally reflecting right-angled isosceles prism may, with advantage, be used as the reflector. One of the smaller faces of the prism rests upon a levelled plane glass plate which is supported above the collimating mirror. The plane of the vertical face of the prism must cut UV at right angles. It will be easily seen that any bending, on entering or leaving the prism, of those rays which are normal to the collimating mirror introduces no error.

81. Practical example. In an experiment by G F. C. Searle and Sub-Lieut C. T. R. Searle, the radii of the steel spheres were $a = \frac{7}{8}$ and $b = \frac{7}{16}$ inch. Hence $n = 0{\cdot}5$. The masses were $m_1 = 358\ 85$ and $m_2 = 44{\cdot}85$ grm. Since $m_1 n^3 = 44{\cdot}85625$, the spheres may be considered to have equal densities. The mass of the solder, 0·1 grm., is negligible. The angle β was measured by a goniometer (§ 35) in which $p = 40{\cdot}00$ cm. The displacement, from its central zero, of the index wire of the goniometer along its scale is denoted by x. The observations were as follows.

x	$\tan\beta$	β	$\sin\beta$	T_0	$T_0^2 \sin\beta$
2·12 cm.	·05300	3° 2′ 2″	·05293	2·543 sec.	·3422
2 56	·06400	3 39 43	·06387	2·318	·3432
3·03	·07575	4 19 55	·07554	2·146	·3481
3·58	·08950	5 6 51	·08914	1·971	·3463

The mean value of $T_0^2 \sin\beta$ is ·3449 sec.². Since $n = 0{\cdot}5$, we have, by (12), $c = 3{\cdot}843 \times \frac{7}{8}$ inch $= 8{\cdot}541$ cm. Hence

$$g = \frac{4\pi^2 c}{T_0^2 \sin\beta} = \frac{4\pi^2 \times 8{\cdot}541}{{\cdot}3449} = 977{\cdot}6 \text{ cm. sec.}^{-2}.$$

NOTE I

On the Vibration of a Body suspended from a Light Spring

In § 30 an approximate solution of the problem has been obtained by means of Lord Rayleigh's theorem. We now give a method of obtaining the solution to any accuracy required.

Let the mass of the spring be S gm., let its unstretched length be l cm. and let S/l be denoted by ρ; the spring is supposed to be uniform. Let the load suspended from the spring be M gm. Let E dynes be the "modulus" of the spring, i.e. the force required to double its length.

Let P, Q (Fig. 33) be two points on the spring such that, when the spring is unstretched, their distances from the fixed end A are x and $x+dx$. Suppose that, under the action of the mass M and of the spring itself, the distance x becomes $x+\xi$. Then the distance $x+dx$ becomes $x+dx+\xi+(d\xi/dx)\,dx$, and thus the length of the element changes from dx to $(1+d\xi/dx)\,dx$. The tension, T, of the spring at this point is therefore

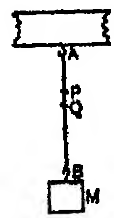

Fig. 33.

$$T=E\frac{d\xi}{dx}. \qquad\qquad (1)$$

The tension at the point determined by $x+dx$ is

$$T+\frac{dT}{dx}dx=E\left(\frac{d\xi}{dx}+\frac{d^2\xi}{dx^2}dx\right), \qquad\qquad (2)$$

and hence the resultant of the two tensions is a force

$$E\frac{d^2\xi}{dx^2}dx$$

in the direction in which x increases, i.e. downwards. The mass of the element is $\rho\,dx$ and hence the equation of motion is

$$\rho\,dx\,.\,\frac{d^2\xi}{dt^2}=E\frac{d^2\xi}{dx^2}dx+g\rho\,dx,$$

or

$$\rho\frac{d^2\xi}{dt^2}=E\frac{d^2\xi}{dx^2}+g\rho. \qquad\qquad (3)$$

We can show that, when the constants a, b, p, q are properly chosen, the equation

$$\xi=ax+\tfrac{1}{2}bx^2+c\cos pt\sin qx \qquad\qquad (4)$$

is a solution of (3).

There is no constant term, since $\xi=0$ when $x=0$, i.e. at the fixed end. Using this value of ξ in (3), we obtain

$$-\rho\,.\,cp^2\cos pt\sin qx=g\rho+E(b-cq^2\cos pt\sin qx).$$

Since this must hold for all values of t, we have

$$Eq^2=p^2\rho, \qquad\qquad (5)$$

$$Eb=-g\rho=-gS/l. \qquad\qquad (6)$$

Equations (5) and (6) determine the values of b and of the ratio p/q. The equation of motion of the mass M is

$$M\frac{d^2\xi}{dt^2}=Mg-T=Mg-E\frac{d\xi}{dx},$$

the quantities having the values corresponding to $x=l$ Hence, by (4),

$$-Mcp^2\cos pt\sin ql=Mg-E\{a+bl+cq\cos pt\cos ql\}.$$

Since this equation holds for all values of t, the constant terms and those involving t must vanish separately. Hence

$$Mg=E(a+bl) \qquad\qquad (7)$$

and

$$Mp^2\sin ql=Eq\cos ql. \qquad\qquad (8)$$

Putting $\theta=ql$, we have, by (8) and (5),

$$\theta\tan\theta=\frac{\rho l}{M}=\frac{S}{M}=K. \qquad\qquad (9)$$

This is the equation from which p is to be found. There are an infinite number of solutions and consequently an infinite number of periodic times. The smallest value of p corresponds to the vibration of longest period.

The steady depression due to a load M is given by $\xi=al+\frac{1}{2}bl^2$, where b is given by (6) and depends only upon the spring. Using (6) and (7) we find

$$\xi=\frac{gl(M+\frac{1}{2}S)}{E}. \qquad\qquad (10)$$

If an additional load m increases the depression from ξ to $\xi+h$, we have, by (10), $h=glm/E$ or

$$E=\frac{glm}{h}. \qquad\qquad (11)$$

When S/M or K is small, the equation $\theta\tan\theta=K$ can be solved by approximation. When θ is small

$$\tan\theta=\theta+\frac{\theta^3}{3}+\frac{2\theta^5}{15}+\frac{17\theta^7}{315}+\dots,$$

so that, by (9),

$$\theta\tan\theta=\theta^2+\frac{\theta^4}{3}+\frac{2\theta^6}{15}+\dots=K$$

The first approximation gives $\theta^2=K$. Making use of this in the term $\theta^4/3$, we have as a second approximation

$$\theta^2=K-\tfrac{1}{3}\theta^4=K-\tfrac{1}{3}K^2.$$

Using this value in the term $\theta^4/3$ and the value K in the term $2\theta^6/15$, and rejecting powers of K above the third, we obtain as a third approximation

$$\theta^2 = K - \tfrac{1}{3}\theta^4 - \tfrac{2}{15}\theta^6 = K - \tfrac{1}{3}(K - \tfrac{1}{3}K^2)^2 - \tfrac{2}{15}K^3$$
$$= K - \tfrac{1}{3}K^2 + \tfrac{4}{45}K^3.$$

Now, if the periodic time be T, we have, by (5) and (11), since $q = \theta/l$ and $p = S/l$,

$$T = \frac{2\pi}{p} = \frac{2\pi l}{\theta}\sqrt{\frac{\rho}{E}} = \frac{2\pi}{\theta}\sqrt{\frac{Sh}{gm}}. \quad \text{.................(12)}$$

Using the value found for θ^2, we have, since $K = S/M$,

$$T = 2\pi\{1 - \tfrac{1}{3}K + \tfrac{4}{45}K^2\}^{-\frac{1}{2}}.\sqrt{\frac{Sh}{Kgm}} = 2\pi\{1 - \tfrac{1}{3}K + \tfrac{4}{45}K^2\}^{-\frac{1}{2}}.\sqrt{\frac{Mh}{gm}}.$$

But
$$(1 - \tfrac{1}{3}K + \tfrac{4}{45}K^2)^{-1} = 1 + \tfrac{1}{3}K + \tfrac{1}{45}K^2 + \ldots,$$

and hence

$$T = 2\pi\sqrt{\frac{(1 + \frac{1}{3}K + \frac{1}{45}K^2)Mh}{gm}} = 2\pi\sqrt{\frac{(M + \frac{1}{3}S + \frac{1}{45}S^2/M)h}{gm}}. \quad \text{...(13)}$$

It will thus be seen that the result obtained by Lord Rayleigh's method in § 30 is a *second* approximation.

The other values of θ will be a little greater than π, 2π, ... Thus we may put $\theta_n = n\pi + \phi_n$, where ϕ_n is small. Then, since $\tan\theta_n = \tan\phi_n$, we have

$$S/M = (n\pi + \phi_n)\tan\phi_n = (n\pi + \phi_n)(\phi_n + \tfrac{1}{3}\phi_n^3 + \ldots) = n\pi\phi_n + \phi_n^2 + \ldots.$$

To the first approximation $\phi_n = S/n\pi M$. Hence, we have, to this approximation,

$$\theta_n = n\pi + \phi_n = n\pi(1 + S/n^2\pi^2 M),$$

and thus, if T_n be the corresponding periodic time,

$$T_n = \frac{2\pi l}{\theta}\sqrt{\frac{\rho}{E}} = \frac{2l}{nv}(1 + S/n^2\pi^2 M)^{-1}, \quad \text{.................(14)}$$

where $\sqrt{E/\rho} = v$, the velocity of a longitudinal wave along the unstretched spring

If the lower end of the spring be *fixed* in any position, the value of ξ for $x = l$ will be independent of the time. Hence we find from (4) that $ql = n\pi$. If T_n' be the corresponding periodic time, we have, by (5),

$$T_n' = \frac{2\pi}{p} = \frac{2l}{n}\sqrt{\frac{\rho}{E}} = \frac{2l}{nv}. \quad \text{..........................(15)}$$

Hence the periodic times given by (14) are all slightly less than the corresponding periodic times when the lower end of the spring is *fixed*.

NOTE II

Periodic Time of a Pendulum vibrating through a Finite Arc

Let O (Fig. 34) be the point of suspension of the simple pendulum OP, and A, A' the extreme positions of P on the two sides of the vertical OC. Let $AOC = A'OC = a$ and let $POC = \theta$. Let the length of the pendulum be l cm. and let the periodic time when the amplitude AOC is infinitesimal be T_0 secs. Then, by § 24,

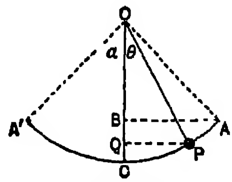

Fig. 34.

$$T_0 = 2\pi\sqrt{l/g}.$$

We have now to calculate the periodic time T when the amplitude is finite.

Since the bob is at rest at A, its velocity, v, at P is that due to its descent from A to P and is the same as that due to a fall from B to Q. Hence

$$v^2 = (l\,d\theta/dt)^2 = 2gl(\cos\theta - \cos a) = 4gl(\sin^2\tfrac{1}{2}a - \sin^2\tfrac{1}{2}\theta)$$

Thus
$$dt = \tfrac{1}{2}\sqrt{\frac{l}{g}}\cdot\frac{d\theta}{\sqrt{\sin^2\frac{1}{2}a - \sin^2\frac{1}{2}\theta}}.$$

Now take a new variable ϕ, where $\sin\frac{1}{2}\theta = \sin\frac{1}{2}a\sin\phi$. Then, when $\theta = 0$, $\phi = 0$, and when $\theta = a$, $\phi = \frac{1}{2}\pi$. Also

$$\sin^2\tfrac{1}{2}a - \sin^2\tfrac{1}{2}\theta = \sin^2\tfrac{1}{2}a\cos^2\phi,$$

and
$$\tfrac{1}{2}\cos\tfrac{1}{2}\theta\,.\,d\theta = \sin\tfrac{1}{2}a\cos\phi\;d\phi.$$

Thus
$$dt = \tfrac{1}{2}\sqrt{\frac{l}{g}}\;\frac{2\sin\frac{1}{2}a\cos\phi\,d\phi/\cos\frac{1}{2}\theta}{\sin\frac{1}{2}a\cos\phi} = \sqrt{\frac{l}{g}}\cdot\frac{d\phi}{\cos\frac{1}{2}\theta}$$
$$= \sqrt{\frac{l}{g}}\cdot\frac{d\phi}{\sqrt{1-\sin^2\frac{1}{2}a\sin^2\phi}}.$$

If T be the time of a complete vibration, T is four times the time taken in passing from C to A or from A to C. While the bob goes from C to A, θ goes from 0 to a and therefore ϕ goes from 0 to $\frac{1}{2}\pi$. Hence

$$T = 4\sqrt{\frac{l}{g}}\cdot\int_0^{\pi/2}\frac{d\phi}{\sqrt{1-\sin^2\frac{1}{2}a\sin^2\phi}} = 4\sqrt{\frac{l}{g}}\,.\,F(\sin\tfrac{1}{2}a),$$

where
$$F(k) = \int_0^{\pi/2}\frac{d\phi}{\sqrt{1-k^2\sin^2\phi}}.$$

Thus
$$T = 2\pi\sqrt{\frac{l}{g}}\cdot\frac{2F(\sin\frac{1}{2}a)}{\pi}.$$

The quantity $F(k)$ is a function of k only and is called the First Complete Elliptic Integral; its value ranges from $\frac{1}{2}\pi$ when k is zero to infinity when k is unity. The values of $F(k)$ were calculated by Legendre for successive values of γ where $\sin\gamma = k$, and a table giving $F(\sin\gamma)$ for each degree from

$\gamma=0°$ to $\gamma=90°$ is given in Dale's Five Figure Tables of mathematical functions

To illustrate the use of the table, we will find T when $a=\frac{1}{2}\pi$, so that the pendulum swings through two right angles. Then

$$\tfrac{1}{2}a=\tfrac{1}{4}\pi \quad \text{and} \quad \sin\gamma=\sin\tfrac{1}{2}a=\sin 45°.$$

The table gives $F(\sin\gamma)=1{\cdot}85407$ for $\gamma=45°$, while $1{\cdot}85407/\frac{1}{2}\pi=1{\cdot}18034$.

Hence
$$T=4\sqrt{\frac{l}{g}}\,F(\sin\tfrac{1}{2}a)=2\pi\sqrt{\frac{l}{g}}\times 1{\cdot}18034,$$

so that the periodic time is about 18 per cent greater than for infinitesimal arcs.

When k is small compared with unity, the integral $F(k)$ can be evaluated in series form, since the series obtained by expanding the square root by the binomial theorem is *uniformly convergent* for all values of k^2 less than unity and can therefore be integrated term by term. Expanding the square root and using the result

$$\int_0^{\pi/2}\sin^{2n}\phi\,d\phi=\frac{2n-1}{2n}\cdot\frac{2n-3}{2n-2}\cdots\frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2},$$

where n is a positive integer, we obtain

$$F(k)=\int_0^{\pi/2}\left\{1+\frac{1}{2}k^2\sin^2\phi+\frac{1\,.\,3}{2\,.\,4}k^4\sin^4\phi+\ldots\right\}d\phi$$
$$=\left\{1+\left(\frac{1}{2}\right)^2k^2+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2k^4+\left(\frac{1\,.\,3\,.\,5}{2\,.\,4\,.\,6}\right)^2k^6+\ldots\right\}\frac{\pi}{2}.$$

Hence

$$T=2\pi\sqrt{\frac{l}{g}}\,\frac{2F(\sin\frac{1}{2}a)}{\pi}=T_0\left\{1+\left(\frac{1}{2}\right)^2\sin^2\tfrac{1}{2}a+\left(\frac{1\,.\,3}{2\,.\,4}\right)^2\sin^4\tfrac{1}{2}a+\ldots\right\}.$$

When a is small, it may suffice to use the expression

$$T=T_0\,(1+a^2/16).$$

The following table, where a is given in degrees, shows the effect of amplitude on the periodic time. Here T_a is the periodic time when the pendulum swings through an angle a on either side of its position of rest, so that the whole arc of vibration is $2a$.

a	T_a/T_0	a	T_a/T_0	a	T_a/T_0
°		°		°	
5	1·00048	50	1·04978	110	1·29534
10	1·00191	55	1·06083	120	1·37288
15	1·00430	60	1·07318	130	1·46982
20	1·00767	65	1·08692	140	1·59445
25	1·01203	70	1·10214	150	1·76220
30	1·01741	80	1·13749	160	2·00751
35	1·02383	90	1·18034	170	2·43936
40	1·03134	100	1·23223	180	infinite
45	1·03997				

NOTE III

Periodic Time for Finite Motion

It is sometimes useful to be able to apply a correction for the amplitude of the vibrations in those cases where the vibrations are not strictly isochronous. Even if we do not apply the correction, we can estimate how small the amplitude must be if the actual periodic time is not to differ from the periodic time for an infinitesimal amplitude by more than—say—one part in a thousand. We shall consider only the case in which the system has only one degree of freedom, so that its configuration is completely fixed by a single coördinate x, which will have the value zero in the equilibrium position of the system The kinetic energy of the system will be proportional to $(dx/dt)^2$, but it may also depend upon x itself. In those cases in which the kinetic energy for a given value of $(dx/dt)^2$ has the same value for $-x$ as for $+x$, we shall be able to write, when x is small enough,

$$\text{Kinetic energy} = A\left(\frac{dx}{dt}\right)^2(1+px^2), \quad \ldots\ldots(1)$$

where A and p are constants

Let the potential energy of the system be $F(x)$, where $F(x)$ is some function of x. Then, if the extreme values of x are $+a$ and $-a$, it follows from the principle of energy that the kinetic energy will be equal to $F(a)-F(x)$, since the system is at rest and has no kinetic energy when $x=a$. When the system is symmetrical with respect to $x=0$, we shall find that, if $F(x)$ be expanded in powers of x,

$$F(x)=B(x^2+qx^4+\ldots), \quad \ldots\ldots(2)$$

where B and q are constants. There are no terms involving x, x^3, ... since $F(x)=F(-x)$, and there is no constant term, if we count the potential energy zero in the equilibrium position. Hence, for small values of a and x,

$$F(a)-F(x)=B\{a^2-x^2+q(a^4-x^4)\}=B(a^2-x^2)\{1+q(a^2+x^2)\}. \quad \ldots(3)$$

Equating expressions (1) and (3) for the kinetic energy, we have

$$A\left(\frac{dx}{dt}\right)^2(1+px^2)=B(a^2-x^2)\{1+q(a^2+x^2)\}.$$

Hence
$$\left(\frac{dt}{dx}\right)^2=\frac{A(1+px^2)}{B(a^2-x^2)\{1+q(a^2+x^2)\}},$$

and thus, if T_a be the periodic time when the system swings between $x=a$ and $x=-a$,

$$T_a=4\sqrt{\frac{A}{B}}\int_0^a\frac{(1+px^2)^{\frac{1}{2}}\,dx}{(a^2-x^2)^{\frac{1}{2}}\{1+q(a^2+x^2)\}^{\frac{1}{2}}}. \quad \ldots\ldots(4)$$

Expanding the expressions involving p and q and neglecting squares and products of p and q, we have

$$T_a=4\sqrt{\frac{A}{B}}\int_0^a\frac{\{1+\frac{1}{2}px^2-\frac{1}{2}q(a^2+x^2)\}\,dx}{(a^2-x^2)^{\frac{1}{2}}}$$

$$=4\sqrt{\frac{A}{B}}\int_0^a\frac{\{1-\frac{1}{2}(2q-p)a^2+\frac{1}{2}(q-p)(a^2-x^2)\}\,dx}{(a^2-x^2)^{\frac{1}{2}}}.$$

Now
$$\int_0^a\frac{dx}{(a^2-x^2)^{\frac{1}{2}}}=\left[\sin^{-1}\frac{x}{a}\right]_0^a=\frac{\pi}{2},$$

$$\int_0^a(a^2-x^2)^{\frac{1}{2}}\,dx=\left[\tfrac{1}{2}x(a^2-x^2)^{\frac{1}{2}}+\tfrac{1}{2}a^2\sin^{-1}\frac{x}{a}\right]_0^a=\tfrac{1}{2}a^2\ \frac{\pi}{2}.$$

Hence, we have

$$T_a=2\pi\sqrt{\frac{A}{B}}\{1-\tfrac{1}{2}(2q-p)a^2+\tfrac{1}{2}(q-p)\cdot\tfrac{1}{2}a^2\}$$

$$=T_0\{1+\tfrac{1}{4}a^2(p-3q)\}, \qquad (5)$$

where T_0 is the periodic time when a is infinitesimal.

Application to pendulum. In the case of a simple pendulum of length l, the kinetic energy is $\frac{1}{2}ml^2(d\theta/dt)^2$, where θ, the angle between the string and the vertical, is the coördinate fixing the configuration of the system. Comparing this value with (1) we see that $p=0$.

Taking the potential energy, V, as zero at the lowest point of the path of the bob, we have $V=mgl(1-\cos\theta)$. Expanding this as far as θ^4, we have

$$V=mgl(\tfrac{1}{2}\theta^2-\tfrac{1}{24}\theta^4)=\tfrac{1}{2}mgl(\theta^2-\tfrac{1}{12}\theta^4).$$

Comparing this with (2) we see that $q=-\frac{1}{12}$.

Hence, by (5), $$T_a=T_0(1+\tfrac{1}{16}a^2),$$

which agrees, as far as a^2, with the result obtained in Note II.

Application to rolling rod. In § 68 we found for the kinetic energy of the rolling rod

$$\tfrac{1}{2}m(k^2+h^2+r^2\theta^2)\,\omega^2=\tfrac{1}{2}m(k^2+h^2)\left(1+\frac{r^2\theta^2}{k^2+h^2}\right)\left(\frac{d\theta}{dt}\right)^2,$$

since $\omega=d\theta/dt$ Comparing this result with (1) we find

$$p=\frac{r^2}{k^2+h^2}.$$

We found for the potential energy

$$V=mg\{(r+h)(\cos\theta-1)+r\theta\sin\theta\}.$$

Expanding this as far as θ^4, we obtain

$$V=mg\{\tfrac{1}{2}(r-h)\theta^2-\tfrac{1}{24}(3r-h)\theta^4\}=\tfrac{1}{2}mg(r-h)\left\{\theta^2-\frac{3r-h}{12(r-h)}\theta^4\right\}.$$

Comparing this with (2) we find

$$q=-\frac{3r-h}{12(r-h)}.$$

Hence, by (5), $$T_a=T_0\left\{1+\tfrac{1}{4}a^2\left(\frac{r^2}{k^2+h^2}+\frac{3r-h}{4(r-h)}\right)\right\}.$$

NOTE IV

Periodic Times of a Pendulum with Two Degrees of Freedom

If the suspending wire OH (Fig. 35) be rigid and if it be hinged to the sphere at a point H on the surface of the sphere, whose centre of gravity is A, the system has two degrees of freedom, when the oscillations are restricted to take place in one plane The periodic times of the two normal vibrations can be found, for small amplitudes, by the method of § 71. We shall consider the case in which the wire OH is massless and of length $l-a$, and the distance AH of the centre of gravity of the suspended body from H is equal to a. The moment of inertia of the body, of mass m, about an axis through A perpendicular to the plane of vibration is mk^2. Let the angle between OH and the vertical OZ be θ; let the angle between HA and the vertical be ϕ, this angle being counted *positive* when OH and HA are deflected in the *same* direction from OZ. Then the angle AHP between AH and the straight line OHP is $\phi-\theta$.

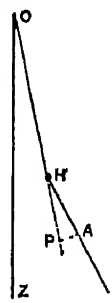

Fig. 35.

One equation of motion is obtained from the theorem that the acceleration of the centre of gravity A is the same as if all the forces acted at A. The acceleration of H in the positive direction at right angles to OH is

$$(l-a)\,(d^2\theta/dt^2)$$

The acceleration of A relative to H has the components $a\,d^2\phi/dt^2$ perpendicular to HA and $a\,(d\phi/dt)^2$ along AH towards H. Hence the acceleration of A perpendicular to OH in the positive direction is

$$l-a)\,(d^2\theta/dt^2)+a\,(d^2\phi/dt^2)\cos(\phi-\theta)-a\,(d\phi/dt)^2\sin(\phi-\theta).$$

When θ and ϕ are always infinitesimal, we may put $\cos(\phi-\theta)=1$ in this expression and may neglect $(d\phi/dt)^2$.

The force acting on the body at right angles to OH is $-mg\sin\theta$ in the positive direction. When θ is infinitesimal, we may put $\sin\theta=\theta$. Then we have to these approximations

$$m\,(l-a)\,\frac{d^2\theta}{dt^2}+ma\,\frac{d^2\phi}{dt^2}=-mg\theta. \quad \ldots\ldots\ldots\ldots\ldots\ldots (1)$$

Again, the angular acceleration of the body about any axis through the centre of gravity is the same as if the centre of gravity were fixed. Since the perpendicular AP on OHP is $a\sin(\phi-\theta)$, the moment about A in the positive direction of the forces acting on the body is $-Ta\sin(\phi-\theta)$, where T is the tension of the wire. But, when the angles θ and ϕ are always very small, the tension will not differ sensibly from mg, and also $\sin(\phi-\theta)$ may be replaced by $\phi-\theta$. Hence we find

$$mk^2\frac{d^2\phi}{dt^2}=-mga\,(\phi-\theta). \quad \ldots\ldots\ldots\ldots\ldots\ldots (2)$$

We can write (1) and (2), if $D^2 = d^2/dt^2$,

$$\{(l-a)D^2+g\}\theta + aD^2\phi = 0 \quad \text{.........................(3)}$$

$$-ga\theta + \{k^2D^2+ga\}\phi = 0. \quad \text{.........................(4)}$$

Hence $$\{(l-a)D^2+g\}\{k^2D^2+ga\}\theta + ga^2D^2\theta = 0. \quad \text{..................(5)}$$

If $\theta = E\sin pt$, $D^2\theta = d^2\theta/dt^2 = -p^2\theta$, and $D^4\theta = -p^2D^2\theta = p^4\theta$. Then, dividing by $p^4\theta$, we find from (5)

$$\frac{ag^2}{p^4} - \frac{g}{p^2}(k^2+al) + (l-a)k^2 = 0. \quad \text{.........................(6)}$$

If the periodic time be T, we have $T = 2\pi/p$, and also $g/p^2 = gT^2/4\pi^2 = \lambda$, where λ is the length of the equivalent pendulum, and thus, from (6), we obtain for the two periodic times

$$gT^2/4\pi^2 = \frac{1}{2a}\{k^2+al \pm \sqrt{(k^2+al)^2 - 4a(l-a)k^2}\}. \quad \text{............(7)}$$

When the body is a uniform sphere of radius a, $k^2 = \frac{2}{5}a^2$, and then

$$T^2 = \frac{4\pi^2}{g}\cdot\frac{1}{10}\{5l+2a \pm \sqrt{25l^2-20la+44a^2}\} \quad \text{...............(8)}$$

$$= \frac{4\pi^2}{10g}\{5l+2a \pm \sqrt{(5l-2a)^2+40a^2}\}. \quad \text{...................(9)}$$

When a/l is small, we obtain, if T_1^2 correspond to the *positive* sign,

$$T_1^2 = \frac{4\pi^2 l}{g}\left(1+\frac{2}{5}\frac{a^2}{l^2}+\frac{4}{25}\frac{a^3}{l^3}+\dots\right), \quad \text{..................(10)}$$

$$T_2^2 = \frac{4\pi^2 . 2a}{5g}\left(1-\frac{a}{l}-\frac{2}{5}\frac{a^2}{l^2}-\dots\right). \quad \text{...............(11)}$$

The relation of ϕ to θ for each mode of vibration can be investigated as in § 71. Since for each mode ϕ has the same frequency as θ, we have $D^2\phi = -p^2\phi$, and then we find, by (4), since $k^2 = \frac{2}{5}a^2$ that

$$\phi = \frac{5\lambda}{5\lambda - 2a}\theta. \quad \text{.................................(12)}$$

By (6) or (8) we find that $\lambda_1\lambda_2 = \frac{2}{5}(l-a)a$ and thus $\lambda_1\lambda_2$ is positive. Since, by (9), λ_1 is positive, it follows that λ_2 is also positive. We then find, by (9), that in the first mode of vibration ϕ and θ have the same sign and that in the second mode they have opposite signs.

In § 25, it was found that, when there was a *rigid* connexion between the sphere and the (massless) wire,

$$T^2 = \frac{4\pi^2 l}{g}\left(1+\frac{2}{5}\frac{a^2}{l^2}\right), \quad \text{.........................(13)}$$

so that, when a/l is small, the value given by (10) is only slightly greater than that given by (13). The latter must be less than the former on account of the constraint due to making the connexion at H rigid.

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